# Stochastic Financial Models – Example sheet 4

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**Problem 1.** Show that the Black–Scholes price of a European call option is strictly convex in both the strike price $K$ and the initial stock price $S_0$, decreasing in $K$ and increasing in $S_0$. Show that the price increases with the interest rate $r$, and with the expiry $T$.

What are the corresponding statements for the Black–Scholes price of a European put option?

**Problem 2.** Consider a stock which pays a dividend at constant rate $\delta \geq 0$. The price of the stock is modelled by

$$S_t = S_0 e^{\sigma W_t + (r-\delta-\sigma^2/2)t}.$$

where $r \geq 0$ is the risk-free interest rate. for a Brownian motion $W$. Show that

$$e^{-rt}S_t + \int_0^t e^{-ru} S_u \delta du$$

defines a martingale. Why then is $S$ a sensible model for the stock price (at least under an equivalent measure)? Show that the time-0 value of a European put option with strike $K$ and expiry $T$ written on this asset is

$$Ke^{-rT}\Phi\left(\frac{\log(K/S_0) - (r-\delta-\sigma^2/2)T}{\sigma\sqrt{T}}\right) - S_0 e^{-\delta T}\Phi\left(\frac{\log(K/S_0) - (r-\delta+\sigma^2/2)T}{\sigma\sqrt{T}}\right).$$

Can we deduce the price of the European call by put-call parity in the case when the stock pays dividends?

**Problem 3.** Show that the joint moment generating function of a Brownian motion and its maximum is given by

$$\mathbb{E}(e^{aW_t + b\max_{s\in[0,t]} W_s}) = \frac{2}{2a+b}\left((a+b)\Phi[(a+b)\sqrt{t}]e^{(a+b)^2t/2} + a\Phi(-a\sqrt{t})e^{a^2t/2}\right)$$

**Problem 4.** A European lookback call option entitles the holder to buy one unit of stock at the expiry time $T$ at the lowest price reached by the stock during the life of the option. Thus, if it is purchased at time 0, at time $T$ it pays off the amount $S_T - \inf_{0\leq u\leq T} S_u$. Find the initial price of such an option in the Black-Scholes model.

**Problem 5.** Let $\mathrm{EC}(S_0, K, \sigma, r, T)$ denote the Black–Scholes price of a European call option with strike $K$, expiry $T$ on an asset with initial price $S_0$, volatility $\sigma$, when the constant interest rate is $r$. Show that the price of a down-and-out call with strike $K$ and a barrier at $B$, where $B < \min\{S_0, K\}$, can be expressed in terms of $EC$ as

$$\mathrm{EC}(S_0, K, \sigma, r, T) - (B/S_0)^{2r/\sigma^2-1}\mathrm{EC}(B^2/S_0, K, \sigma, r, T).$$

**Problem 6.** In the Black–Scholes model, find the time-0 prices of European contingent claims which pay at time $T$ the amounts:

(a) $\int_0^T S_u du$

(b) $(\log S_T)^2$.

**Problem 7.** Given times $0 < T_0 < T_1$, a forward start call option gives the right (but not the obligation) to buy a certain stock at time $T_1$ at the stock's price at time $T_0$. Explain why the payout is $(S_{T_1} - S_{T_0})^+$ and find its initial price in the Black–Scholes model. How is this option hedged?

**Problem 8.** The Black–Scholes PDE is given by

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 V}{\partial s^2} + rs\frac{\partial V}{\partial s} - rV = 0.$$

(i) Set $v(\tau, x) = V(T - 2\tau/\sigma^2, e^x)$. Show that

$$\frac{\partial v}{\partial \tau} - \frac{\partial^2 v}{\partial x^2} + (1-k)\frac{\partial v}{\partial x} + kv = 0$$

for constant $k$ which you should determine.

(ii) Let $\alpha, \beta$ be real constants and defined $u(x, \tau)$ via

$$u(\tau, x) = e^{\alpha x + \beta \tau} v(\tau, x).$$

Find the PDE for $u(x, \tau)$ and choose $\alpha, \beta$ such that $u$ satisfies the standard heat equation,

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}.$$

**Problem 9.** Consider the standard heat equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}.$$

Assume $t \in [0, T]$, $x \in [0, L]$ and inital/boundary data

$$u(0, x) = g(x),\ u(t, 0) = a(t),\ u(t, L) = b(t).$$

The grid $\{(i\delta_t, j\delta_x) : i = 1, ..., N_t, j = 1, ..., N_x\}$ with $\delta_x = L/N_x, \delta_t = T/N_t$ is given and we seek approximations $U_j^i \approx u(i\delta_t, j\delta_x)$. Set $U^i = \left(U_j^i\right)_{j=1,...,N_x-1} \in \mathbb{R}^{N_x-1}$.

(i) Formulate the FTCS-method as linear equations

$$U^{i+1} = FU^i + p^i$$

for some $(N_x - 1) \times (N_x - 1)$-matrix $F$ and $p^i \in \mathbb{R}^{N_x-1}$.

(ii) Similarly, formulate the BTCS-method as

$$BU^{i+1} = U^i + q^i$$

for a matrix $B$ and a vector $q^i$ to be determined.

(iii) Show that adding these two linear equations yields exactly the Crank–Nicolson method.