Stochastic Financial Models – Example sheet 3

Lent 2017, SA

Problem 1. Consider the binomial model in the case $(1+a)(1+b)=1$. Find the arbitrage-free price of the following contingent claims.

(i) An up-and-out call option with payoff-function

$$C_{\text{uo}} := \begin{cases} 0 & \text{if } \max_{0\le t\le T} S_t^1 \ge B, \\ (S_T^1 - K)^+ & \text{otherwise,} \end{cases}$$

where $K > 0$ is the strike price and $B > S_0^1 \vee K$ is an upper barrier for the stock price.

(ii) The lookback put option with payoff function

$$C_{\text{lp}} := \max_{0\le t\le T} S_t^1 - S_T^1.$$

(iii) The lookback call option with payoff function

$$C_{\text{lc}} := S_T^1 - \min_{0\le t\le T} S_t^1.$$

[Hint: For (i) find a relation between the up-and-out call, the up-and-in call C_{uo} considered in the lectures and the corresponding 'plain vanilla call' with payoff $EC(K,T) = (S_T^1 - K)^+$. For (ii) and (iii) use the reflection principle.]

Problem 2. An investor with wealth X_0 at time 0 wishes to invest it in such as way as to maximise $\mathbb{E}[U(X_N)]$, where X_N is the wealth at the start of day N, and $U(x) = \frac{1}{1-R}x^{1-R}$ for a relative risk aversion coefficient $R > 0$, $R \neq 1$. He chooses the proportion θ_n of his wealth to invest in the single risky asset during the period $(n-1, n]$, so that his wealth at the start of day n will be

$$X_n = X_{n-1}[\theta_n \xi_n + (1-\theta_n)(1+r)]$$

where $\xi_1, \xi_2, \ldots$ are i.i.d. positive random variables, and r is the per-period riskless rate of interest. Find the form of the optimal policy in each of the two situations:

(i) θ_n unrestricted,

(ii) $0 \le \theta_n \le 1$.

Show that the solutions are the same if and only if

$$\frac{\mathbb{E}(\xi^{1-R})}{\mathbb{E}(\xi^{-R})} \le 1 + r \le \mathbb{E}(\xi)$$

where ξ is a random variable with the same distribution as the ξ_i's

[Hint: look for a value function of form $V(n, x) = a_n U(x)$.]

Problem 3. A gambler has the chance to bet on a sequence of N coin tosses. Let $\xi_n = 1$ if the n-th toss is a head, otherwise $\xi_n = -1$ if a tail. The $\xi_1, \xi_2, \ldots$ are independent but *not* identically distributed; $\mathbb{P}(\xi_n = 1) = p_n \geq 1/2$. If the gambler's wealth just before the n-th toss is w_{n-1}, he may stake any amount $x_n \in [0, w_{n-1}]$ on the toss of the coin; his wealth at time n is therefore

$$w_n = w_{n-1} + x_n \xi_n.$$

Determine how the gambler should play so as to maximise his final expected utility $\mathbb{E}[\log w_N]$. Hint: look for a value function of form $V(n, w) = a_n + \log w$.

Problem 4. If $(W_t)_{t \geq 0}$ is a Brownian motion, show that the following processes are martingales:

(i) $W_t^2 - t$.

(ii) $e^{\theta W_t - \theta^2 t/2}$ for any $\theta \in \mathbb{R}$.

(iii) $\cosh(\theta W_t) e^{-\theta^2 t/2}$ for any $\theta \in \mathbb{R}$.

Problem 5. Using a suitable martingale and the optional stopping theorem, show that if $T_a = \inf\{t \geq 0 : W_t \geq a\}$ is the first time that a Brownian motion exceeds level $a > 0$, then the Laplace transform of T_a is given by

$$\mathbb{E}[e^{-\lambda T_a}] = e^{-a\sqrt{2\lambda}}.$$

For the brave of heart: confirm this by integrating the density of T_a as derived from the reflection principle.

Problem 6. An agent holds a single share of a firm that will go bankrupt at time τ, an exponential random variable with mean $1/\lambda$. Suppose that the price of the share at time $0 \leq t < \tau$ is $S_t = S_0 + W_t$, where W is a Brownian motion independent of τ. The agent plans to sell the share at the first time H_b that the price exceeds b, and seeks to maximise

$$\mathbb{E}[b e^{-r H_b} \mathbb{1}_{\{H_b < \tau\}}]$$

Find the optimal choice of b.

Problem 7. Suppose that $a < x < b$, and let $\tau = \inf\{t \geq 0 : W_t \geq b \text{ or } W_t \leq a\}$ Using an appropriate martingale and the optional stopping theorem, prove that

$$\mathbb{E}[e^{-\theta^2 \tau/2} | W_0 = x] = \frac{\cosh[\theta(x - \frac{1}{2}(a+b))]}{\cosh[\frac{1}{2}\theta(b-a)]}$$