Stochastic Financial Models – Example sheet 2

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Problem 1. Suppose that $X_1, X_2, \ldots$ are i.i.d. real random variables with $\mathbb{E}(|X_1|) < \infty$. Let $S_0 = 0$ and $S_n = X_1 + \ldots + X_n$.

(a) When is $(S_n)_{n\geq 0}$ a martingale? Specify the filtration.

(b) Show that
$$\mathbb{E}[X_1|S_n] = \frac{S_n}{n}.$$

(c) Compute $\mathbb{E}[S_n|X_1]$.

(d) Find an example of a process $(Z_n)_{n\geq 0}$ adapted to some filtration which has the property
$$\mathbb{E}[Z_{n+1}|Z_n] = Z_n$$
for all $n \geq 0$, but $\mathbb{E}[Z_{N+1}|\mathcal{F}_N] \neq Z_N$ for some N. [Hint: Use part (b) with $Z_1 = S_1$ and $Z_2 = S_2$ but $Z_3 \neq S_3$]

Problem 2. Consider a (homogenous) Markov-chain $(X_n)_{n\geq 0}$ on a finite state-space S with transition matrix P. A function $f : S \to \mathbb{R}$ is considered as a column vector so that Pf makes sense as matrix multiplication. Let $\mathcal{F}_n = \sigma(X_k : 0 \leq k \leq n)$.

(a) Check that
$$[Pf](X_n) = \mathbb{E}[f(X_{n+1})|\mathcal{F}_n].$$

(b) Fix $f : S \to \mathbb{R}$ define
$$M_n = f(X_n) - f(X_0) - \sum_{k=0}^{n-1}[(P-I)f](X_k).$$

Show that $(M_n)_{n\geq 0}$ is a martingale.

(c) A function $f : S \to \mathbb{R}$ is called subharmonic if $f(x) \leq [Pf](x)$ for all x. Show that $(f(X_n))_{n\geq 0}$ a submartingale if f is subharmonic. (This explains the 'sub' in the definition of submartingale.)

Problem 3.

(a) Given a sigma-algebra $\mathcal{G}$, show that $A \in \mathcal{G}$ if and only if $\mathbb{1}_A$ is $\mathcal{G}$-measurable.

(b) Let τ be a stopping time for the filtration $(\mathcal{F}_n)_{n\geq 0}$ Show that $\mathbb{1}_{\{\tau\geq n+1\}}$ is $\mathcal{F}_n$-measurable for all $n \geq 0$.

(c) Let $M = (M_n)_{n\geq 0}$ be a submartingale and τ a stopping time. Show that the stopped submartingale M^τ defined as

$$M_n^\tau = M_{\tau\wedge n}$$

is still a submartingale.

Problem 4. Let $X_1, X_2, \ldots$ be i.i.d. random variables with $\mathbb{E}(X_1) = \mu$, $\mathrm{Var}(X_1) = \sigma^2$ and moment generating function $\phi(\theta) = \mathbb{E}[e^{\theta X_1}]$, where ϕ is assumed finite valued. Assuming $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$, show that the following are martingales

(a) $M_n = S_n^2 - \sigma^2 n$ if and only if $\mu = 0$.

(b) $N_n = e^{\theta S_n}\phi(\theta)^{-n}$

where $S_n = X_1 + \ldots + X_n$.

Problem 5. Fix $s \in \mathbb{Z}$, and suppose that $X_1, X_2, \ldots$ are i.i.d. random variables with values in $\{-1, 1\}$ so that $\mathbb{P}(X_1 = 1) = p = 1 - \mathbb{P}(X_1 = -1)$ for some fixed $p \in (0, 1)$. Let S be the process defined by $S_0 = s$ and $S_n = S_{n-1} + X_n$, i.e. a simple random walk started at $S_0 = s$.

(a) Show that the processes M and N defined by

$$M_n = \left(\frac{1-p}{p}\right)^{S_n} \text{ and } N_n = S_n + n(1-2p)$$

are martingales with respect to the filtration given by $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$.

Now assume $p = 1/2$, so that S is a simple symmetric random walk.

(b) Suppose $S_0 = 1$.

(i) Show that $\tau = \inf\{n \geq 0, S_n = 0\}$ is a stopping time, possibly taking the value ∞.

(ii) Apply the martingale convergence theorem to see that the stopped martingale S^τ converges almost surely (to what?). Conclude that $\tau < \infty$ a.s.

(iii) Show that the martingale S^τ does not converge in L^1, i.e. $\mathbb{E}(|S_n^\tau - S_\infty^\tau|)$ does not tend to 0 as $n \to \infty$.

(c) Now let S be the simple symmetric random walk started at $S_0 = 0$.

(i) Fix integers $a, b \geq 0$ and let $\tau = \inf\{n \geq 0, S_n = -a \text{ or } S_n = b\}$. Check that τ is a stopping time. Why is $\tau < \infty$ almost surely?

(ii) Use the optional stopping theorem to compute the probability that S hits $-a$ before b. Compute $\mathbb{E}(\tau)$. [Hint: Show that $S_n^2 - n$ defines a martingale, and apply the optional stopping theorem to it.]

Problem 6. At time 1 an urn contains a white and a red ball. Take out a ball at random and replace it by two balls of the same colour; this gives the new content of the urn at time 2. Keep iterating this procedure.

Let Y_n be the number of white balls in the urn at time n, and let $X_n = \frac{Y_n}{n+1}$. Show that X_n is a.s. convergent to a random variable U. Compute the mean of U. Can you compute the variance of U? [Hint: Consider the process $\frac{Y_n(Y_n+1)}{(n+1)(n+2)}$.]

Problem 7. Consider a single-period trinomial model, with two assets, a riskless bond and a risky stock. Suppose that initially both are worth $S_0^0 = S_0^1 = 1$. The riskless rate is r so $S_1^0 = 1+r$. The risky asset at time 1 will be worth a if the period was bad, b if the period was indifferent, and c if the period was good, $a < b < c$, and these are the only possibilities. We assume $a < 1 + r < c$.

(a) Find all the risk-neutral measures for this model.

(b) For simplicity only, assume $r = 0$. Characterise all contingent claims with payout $Y = f(S_1^1)$ at time 1 that can be replicated, that is for which there exists $\bar{\pi} \in \mathbb{R}^2$ such that

$$Y = \bar{\pi} \cdot \bar{S}_1.$$

Determine the price of this contingent claim at time 0. Compute the expectation of Y with respect to any equivalent martingale measure. Conclusion?

(c) How would your analysis extend to a single-period model with $d + 1$ assets?

Problem 8. Consider a one-period binomial model with a stock and a riskless asset, that is S^0 and S^1 are defined as in Problem 7, but at time 1 the risky asset S^1 takes values a and c only. A utility-maximising investor has initial wealth $w_0 > 0$ and utility $U(x) = \sqrt{x}$. Find the agent's optimal investment in the risky stock, and verify it has the same sign as $\mathbb{E}[S_1] - (1 + r)S_0$, where r is the riskless interest rate, and S_t is the price of the stock at time t.

Problem 9. Consider a single-period model with a risky asset S^1 having initial price S_0^1. At time 1 its value S_1^1 is a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ of the form

$$S_1^1 = \exp(\sigma Z + m), \qquad m \in \mathbb{R}, \sigma > 0,$$

where $Z \sim N(0, 1)$. (S_1^1 is then also said to be log-normal distributed). For simplicity assume that there is a riskless asset S^0 with $S_0^0 = S_1^0 = 1$ (so $r = 0$). Find a risk-neutral measure $\mathbb{Q}$ for this model. [Hint: Consider a density of the form $\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp(\tilde{\sigma} Z + \tilde{m})$ and find suitable $\tilde{m}$ and $\tilde{\sigma}$.]