Stochastic Financial Models – Example sheet 1

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Problem 1. Some useful facts about the Gaussian distribution for later reference. Here, Φ denotes the standard Gaussian cumulative distribution function.

(i) Suppose that $X \sim N(\mu, \sigma^2)$. Show that

(a) $\mathbb{E}[e^{\theta X} f(X)] = e^{\mu\theta + \theta^2\sigma^2/2}\mathbb{E}[f(X + \theta\sigma^2)]$ for all $\theta \in \mathbb{R}$ and suitable f,

(b) $\mathbb{E}[f(X)(X - \mu)] = \sigma^2\mathbb{E}[f'(X)]$ for suitable f,

(c) $\mathbb{E}\ \Phi(\alpha X + \beta) = \Phi\left(\frac{\alpha\mu+\beta}{\sqrt{1+\alpha^2\sigma^2}}\right)$ for all $\alpha, \beta \in \mathbb{R}$.

(ii) Suppose (X, Y) have the jointly Gaussian distribution. Show that

(a) $\mathbb{E}[Y - \mathbb{E}(Y)|X] = \frac{\mathrm{Cov}(X,Y)}{\mathrm{Var}(X)}[X - \mathbb{E}(X)]$

(b) $\mathrm{Cov}(f(X), Y) = \mathbb{E}[f'(X)]\mathrm{Cov}(X, Y)$ for suitable f.

Problem 2. Suppose that $U : \mathbb{R} \to \mathbb{R}$ is concave, that the random variable Z has zero mean, and $\mu \in \mathbb{R}$ is a constant. Letting

$$\phi(t) = \mathbb{E}[U(\mu + tZ)]$$

prove that ϕ is concave. Prove also that ϕ is decreasing in $[0, \infty)$ and increasing in $(-\infty, 0]$.

Problem 3. Suppose that U is a concave increasing function, and that $X \sim N(\mu, \sigma^2)$. Writing $\psi(\mu, \sigma) = \mathbb{E}^{(\mu,\sigma)}[U(X)]$, show that ψ is concave: for $0 < p < 1$

$$\psi(p\mu_1 + (1-p)\mu_2, p\sigma_1 + (1-p)\sigma_2) \geq p\psi(\mu_1, \sigma_1) + (1-p)\psi(\mu_2, \sigma_2).$$

Show that $\psi(\mu, \sigma)$ increases with μ and decreases with $\sigma \geq 0$, and hence show in the context of mean-variance analysis that, if all returns are jointly Gaussian, an investor maximizing the expected utility of his final wealth will choose a mean-variance-efficient optimal portfolio.

Problem 4. Suppose that an agent has expected-utility preferences over contingent claims, represented by a concave function U. Suppose further that if two contingent claims have the same mean and the same variance then he will be indifferent between them. By considering suitable distributions concentrated on three points, or otherwise, prove that the function U must be quadratic.

Problem 5. Suppose an agent has 'utility' function

$$U(x) = x - \frac{1}{2}\varepsilon x^2$$

and may invest in a single stock, worth $S_0 = 1$ at time 0 and some random amount S_1 at time 1, and in a bond, worth 1 at both times 0 and 1. Let Y be the random payout at time 1 of some contingent claim. Given initial wealth $X_0 = x$, show that his optimization problem

$$\text{maximise } \mathbb{E}[U(x + \pi(S_1 - 1) + Y)]$$

is solved by taking

$$\pi = \frac{\mathbb{E}[(S_1 - 1)(1 - \varepsilon(x + Y))]}{\varepsilon \mathbb{E}[(S_1 - 1)^2]}$$

Show that his maximised objective is then equal to

$$\mathbb{E}[U(x + Y)] + \frac{1}{2} \frac{\mathbb{E}\big[(S_1 - 1)(1 - \varepsilon(x + Y))\big]^2}{\varepsilon \mathbb{E}[(S_1 - 1)^2]}$$

In the special case where $Y = 0$ this reduces to

$$U(x) + \frac{1}{2}(1 - \varepsilon x)^2 \frac{\mathbb{E}[S_1 - 1]^2}{\varepsilon\, \mathbb{E}[(S_1 - 1)^2]}.$$

Explain briefly how this would allow you to compute indifference prices for non-marketed contingent claims, i.e. claims that do not have already have a market price.

In the next three questions, we consider a single-period model with $d > 1$ risky assets, worth $S_0 = (S_0^1, \ldots, S_0^d)^\top$ at time 0 and $S_1 \sim N(\mu, V)$ at time 1, where $\mu \in \mathbb{R}^d$ is a given mean vector and V is a non-singular $d \times d$ covariance matrix. A riskless asset (asset 0) may be added to this market; the notations $\bar{S}_t = (S_t^0; S_t)$, and $S_1^0 / S_0^0 = 1 + r$ are then used.

Problem 6. (a) An agent aims to maximize his expected utility $\mathbb{E}[U(\theta \cdot S_1)]$ of wealth at time 1, subject to his budget constraint $\theta \cdot S_0 = w_0$. Show that the form of his optimal portfolio is

$$\theta^* = \frac{1}{\gamma} V^{-1} \mu + \frac{\gamma X_0 - S_0^\top V^{-1} \mu}{\gamma S_0^\top V^{-1} S_0} V^{-1} S_0$$

exactly as it would be for the case of CARA utility with a coefficient of absolute risk aversion defined by

$$\gamma = -\frac{\mathbb{E}[U''(X_1^*)]}{\mathbb{E}[U'(X_1^*)]}$$

where $X_1^* = \theta^* \cdot S_1$.

(b) Suppose we now add the riskless asset. Show that with the same interpretation of γ, the investor's optimal portfolio is again exactly as it would be were his utility CARA with coefficient of absolute risk aversion γ.

Problem 7. We have usually thought of the time-0 prices S_0 as given, and computed agents' optimal demands for the assets based on this. However, in an equilibrium analysis, the prices S_0 would be adjusted to clear markets. Suppose there is unit net supply of asset i, for each $i = 1, \dots, d$, and zero net supply of asset 0; the (equilibrium) prices must be such that the total demand of all agents for each risky asset is 1, and for asset 0 is 0. Suppose that there are K agents in the market, agent k having CARA utility with coefficient of absolute risk aversion γ_k, and that agent k enters the market with a portfolio $\alpha_k \in \mathbb{R}^d$ of shares, so that

$$\sum_k \alpha_k = \mathbf{1}.$$

where $\mathbf{1} = (1, \dots, 1)^\top \in \mathbb{R}^d$. Without loss of generality, suppose $S_0^0 = 1$. Show that the market-clearing time-0 prices for the risky assets must be

$$S_0 = \frac{(\mu - \Gamma V \mathbf{1})}{1 + r}$$

where $\Gamma = (\sum_k \gamma_k^{-1})^{-1}$.

Problem 8. For simplicity, suppose that $r = 0$. Suppose now that the agents decide to open a market in further assets $Y = (Y_1, \dots, Y_m)^\top$ which are in zero net supply, and have the property that $(S^\top, Y^\top)^\top$ is multivariate Gaussian, with mean $\bar{\mu}$ and covariance $\bar{V}$. Prove that for this model the time-0 prices of the assets S are unaffected, and that at time 0 none of the agents hold any of the assets Y. Find the time-0 prices of the assets Y.

Problem 9. In a single-period investment model, the two risky assets have intial prices $S_0^1 = 2$ and $S_0^2 = 3$ and the time-1 prices have moments $\mathbb{E}(S_1^1) = 3, \mathbb{E}(S_1^2) = 4, \mathrm{Var}(S_1^1) = 2, \mathrm{Var}(S_1^2) = 3$, $\mathrm{Cov}(S_1^1, S_1^2) = 2$. Find the portfolio which has minimum variance subject to having initial wealth $w_0 = 10$ and mean return $\mathbb{E}(w_1) = 14$. What is the variance of this portfolio? (Assume that there is a riskless asset with risk-free rate $r = 0$. Why is the problem trivial if there is no riskless asset?)