STOCHASTIC FINANCIAL MODELS: Examples 4 (of 4)

1. Show that the price of a European call option is strictly convex in both the strike price and the stock price, decreasing in the first and increasing in the second. Show that the price increases with the interest rate r, and with the expiry T.

   What are the corresponding comparative statics for the put option?

2. Suppose that S is a Black-Scholes asset which pays dividends at constant rate $\delta \geq 0$; in the pricing probability,

$$S_t = S_0 \exp(\sigma B_t + (r - \delta - \frac{1}{2}\sigma^2)t).$$

   Show that the time-0 value of a European put option with strike K and expiry T written on this asset is

$$Ke^{-rT}\Phi(a) - S_0 e^{-\delta T}\Phi(a - \sigma\sqrt{T}),$$

   where

$$a = \frac{\log(K/S_0) - (r - \delta - \sigma^2/2)T}{\sigma\sqrt{T}}.$$

   Can we deduce the price of the European call by put-call parity in the case when the stock pays dividends?

3. A European *lookback* call option entitles the holder to buy one unit of stock at the expiry time T at the lowest price reached by the stock during the life of the option. Thus, if it is purchased at time 0, at time T it pays off the amount $S_T - \inf_{0 \leq u \leq T} S_u$. In the Black-Scholes model show that the price at time 0 of such an option is

$$S_0 \left[\left(\frac{2r + \sigma^2}{2r} \right) \Phi \left(\left(\frac{2r + \sigma^2}{2\sigma/\sqrt{T}} \right) \right) - e^{-rT} \left(\frac{2r - \sigma^2}{2r} \right) \Phi \left(\left(\frac{2r - \sigma^2}{2\sigma/\sqrt{T}} \right) \right) - \frac{\sigma^2}{2r} \right],$$

   if $r \neq 0$.

4. Let $EC(S_0, K, \sigma, r, T)$ denote the (Black-Scholes) price of a European call option with strike K, expiry T on an asset with initial price S_0, volatility σ, when the constant interest rate is r. Show that the price of a down-and-out call with strike K and a barrier at $S_0 e^b < \min\{S_0, K\}$ can be expressed in terms of EC as

$$EC(S_0, K, \sigma, r, T) - e^{2\mu b/\sigma^2} EC(S_0 e^{2b}, K, \sigma, r, T),$$

   where $\mu = r - \frac{1}{2}\sigma^2$.

5. In the Black-Scholes model, find the time-0 prices of European contingent claims which pay at time T the amounts:

$$(a) \int_0^T S_u du; \qquad (b) (\log(S_T))^2.$$

6. The Black-Scholes model can be approximated by a binomial model, but even once we have fixed the step $h > 0$ of the time discretisation there are several possible ways in which this might be done. Assuming that $r \geq 0$, consider the following possibilities:

   (i) Choose $d = 1/u$, so that an up step followed by a down step returns us to the initial value. The probability p of an up step is chosen so as to match the first two moments of the share price move over the time interval of length h. Write down the two equations which determine u and p, and prove that u is the larger root of the quadratic

   $$u^2 - u(e^{-rh} + e^{rh+\sigma^2 h}) + 1 = 0.$$

   Prove that $u > 1$ always, and find the expression for p, proving that always $0 < p < 1$.

   (ii) Choose $d = 1/u = e^{-a}$, and choose p and a so that the first two moments of the log price are matched. In this case, show that

   $$\begin{aligned} a &= \sqrt{\sigma^2 h + (r - \frac{1}{2}\sigma^2)^2 h^2} \\ p &= \frac{(r - \frac{1}{2}\sigma^2)h + a}{2a}. \end{aligned}$$

   (iii) Take $u = \exp(\sigma\sqrt{h} + (r - \frac{1}{2}\sigma^2)h)$, and $d = \exp(-\sigma\sqrt{h} + (r - \frac{1}{2}\sigma^2)h)$ with $p = 1/2$ (this corresponds to approximating the underlying Brownian motion with a simple symmetric random walk).

   (iv) (OPTIONAL) Write a program to compute the price of a European call option with expiry 0.5, volatility $\sigma = 0.2$, riskless rate $r = 0.05$, strike $K = 90$ and initial value $S_0 = 100$, using each of the above three methods with a variety of values of h, and compare with the exact Black-Scholes price.

   By placing the root of the tree at the initial price, we will usually find that the strike price is not at a node of the tree. Modify your code to ensure that the strike price always lies at a node of the tree at expiry, and compare the prices you now get with Black-Scholes. (You will need to think how to evaluate the call price when $S_0 = 100$, as this point will typically not be on the grid.)

7. The Black-Scholes PDE is given by

   $$\partial_t V + \frac{1}{2}\sigma^2 s^2 \partial_{ss} V + rs\partial_s V - rV = 0.$$

   (i) Consider the change of variable $(t, s) \mapsto (\tau, x)$ given by

   $$s = e^x, \ \tau = \frac{1}{2}\sigma^2 (T - t)$$

   and set $v(\tau, x) = V(t, s)$. Show that for

   $$\partial_\tau v - \partial_{xx} v + (1 - k) v_x + kv = 0$$

   for constant k which you should determine.

(ii) Let α, β be real constants and defined $u(\tau, x)$ via

$$v(\tau, x) = e^{\alpha x + \beta \tau} u(\tau, x).$$

Find the PDE for $u(\tau, x)$ and choose α, β such that u satisfies the standard heat equation,

$$\partial_\tau u = \partial_{xx} u.$$

8. Consider the PDE

$$0 = \partial_t u + \tfrac{1}{2} \partial_{xx} u.$$

Assume $t \in [0, T]$, $x \in [0, L]$ and inital/boundary data

$$u(T, x) = g(x), \ u(t, 0) = a(t), \ u(t, L) = b(t).$$

The grid $\{(ik, jh) : j = 1, ..., N_x, i = 1, ..., N_t\}$ with $h = L/N_x, k = T/N_t$ is given and we seek approximations $U_j^i \approx u(ik, jh)$. Set $U^i = \left(U_j^i\right)_{j=1,\ldots,N_x} \in R^{N_x - 1}$.

(i) Formulate the explicit finite-difference scheme as linear equations

$$U^i = FU^{i+1} + p^i$$

for some $(N_x - 1) \times (N_x - 1)$-matrix F and $p^i \in R^{N_x - 1}$.

(ii) Similarly, formulate the fully-implicit finite-difference scheme as

$$BU^i = U^{i+1} + q^i$$

for a matrix B and a vector q^i to be determined.

(iii) Show that adding these two linear equations yields exactly the Crank-Nicolson method.