LCGR

STOCHASTIC FINANCIAL MODELS: Examples 3 (of 4)

1 . An investor with wealth w_0 at time 0 wishes to invest it in such as way as to maximise $\mathbb{E}U(w_N/w_0)$, where w_N is the wealth at the start of day N, and $U(x) = x^{1-R}/(1-R)$. Each day, he chooses the proportion θ of his wealth to invest in the single risky asset, so that his wealth at the start of day $n+1$ will be

$$w_{n+1} = w_n\{\,\theta X_n + (1-\theta)(1+r)\}$$

where the X_i are independent identically-distributed positive random variables, and r is the per-period riskless rate of interest. Find form of the optimal policy in each of the two situations

(i) θ unrestricted;

(ii) $0 \le \theta \le 1$.

Show that the solutions are the same if and only if

$$\frac{\mathbb{E}(X_0^{1-R})}{\mathbb{E}(X_0^{-R})} \le 1 + r \le \mathbb{E}X_0.$$

2 . At the start of year n, an investor receives income $Y_n \ge 0$. He chooses to invest a proportion $\theta_n \in [0,1]$ of this in the stock market, and consumes the remainder. At the start of the next year, his income is

$$Y_{n+1} = Y_n + \theta_n Y_n X_n,$$

where $X_n > 0$ is the return on the stock market in year n. The X_n are independent identically-distributed random variables, with mean μ. The investor's objective is to choose the θ_n so as to maximise

$$\mathbb{E}\Big\{\sum_{n=0}^{N-1}(1-\theta_n)Y_n + Y_N\Big\}.$$

How should he invest so as to achieve this?

3 . A gambler has the chance to bet on a sequence of N coin tosses. Let $\xi_n = 1$ if the n^{th} toss is a Head, and let $\xi_n = -1$ otherwise. The ξ_n are independent but not identically distributed; $\mathbb{P}(\xi_n = 1) = p_n \ge 1/2$. If the gambler's wealth just before the n^{th} toss is w_{n-1}, he may stake any amount $x \in [0, w_{n-1}]$ on the toss of the coin; his wealth at time n is therefore

$$w_n = w_{n-1} + x\xi_n.$$

Determine how the gambler should play so as to maximise his final expected utility $\mathbb{E}\log(w_N)$.

4 . An agent starts with initial capital C. At the beginning of day n ($n = 1, \ldots, N$) he observes a new non-negative random variable Y_n, and then chooses c_n, the amount of his remaining capital to consume that day. The Y_n are independent, $\mathbb{E}Y_n = \mu_n$. His objective is to maximise

$$\mathbb{E}\Big[\sum_{n=1}^{N} Y_n \log(c_n)\Big]$$

subject to the constraint $\sum_{n=1}^{N} c_n = C$. Find his optimal consumption policy, and an expression for the maximal value of his objective.

5 . If $(B_t)_{t\geq 0}$ is a Brownian motion, show that the following processes are martingales:

(i) $B_t^2 - t$;

(ii) $\exp(\theta B_t - \frac{1}{2}\theta^2 t)$ for any $\theta \in \mathbb{R}$;

(iii) $B_t^3 - 3tB_t$;

(iv) $\cosh(\theta B_t)e^{-\theta^2 t/2}$ for any $\theta \in \mathbb{R}$.

6 . An agent holds a single share, whose value at time t is $S_0 + \sigma B_t$. The firm issuing the share may go bankrupt at some exponentially-distributed random time T with mean $1/\lambda$. The agent plans to sell the share at the first time H_a that the price exceeds a; if this comes before T, then the value to the agent is $a\exp(-rH_a)$, otherwise the value is nothing. Find the agent's optimal choice of a.

7 . Using the martingale from the second part of Question 5, and the Optional Sampling Theorem, show that if $H_a \equiv \inf\{t : B_t > a\}$ is the first time that a Brownian motion exceeds level $a > 0$, then the Laplace transform of H_a is given by

$$\mathbb{E}\exp(-\lambda H_a) = \exp(-a\sqrt{2\lambda}).$$

Confirm this by integrating the density of H_a as derived from the Reflection Principle.

8 . Suppose that $a < x < b$, and let $\tau \equiv \inf\{t : B_t \notin [a, b]\}$. For $\theta > 0$, using an appropriate martingale and the Optional Sampling Theorem, prove that

$$\mathbb{E}[\exp(-\frac{1}{2}\theta^2\tau)|B_0 = x] = \frac{\cosh(\theta(x - (a + b)/2))}{\cosh(\theta(b - a)/2)}.$$