STOCHASTIC FINANCIAL MODELS: Examples 2 (of 4)

1. Suppose that $X_1, X_2, ...$ are IID real random variables with $E(|X_1|) < \infty$. Let

$$S_n := X_1 + ... + X_n.$$

   (a) When is (S_n) a martingale? Specify the filtration.

   (b) Show that

$$\mathbb{E}\left[X_1 | S_n\right] = \frac{S_n}{n}.$$

   (c) Compute $\mathbb{E}\left[S_n | X_1\right]$.

   (d) Find an example of a process $(Z_n)_{n\in I}$ adapted to some filtration $(\mathcal{F}_n)_{n\in I}$ which satisfies

$$\mathbb{E}\left[Z_{n+1} | Z_n\right] = Z_n \text{ but } \mathbb{E}\left[Z_{n+1} | \mathcal{F}_n\right] \neq Z_n.$$

   (Hint: Take $X \sim N(0,1)$, $I = \{1,2,3\}$, $Z_1 = X_1, Z_2 = X_1 + X_2$ and construct $Z_3 - Z_2$ independent of Z_2 but fully determined at times 1 and 2.)

2. Consider a (homogenous) Markov-chain $(X_n)_{n\in\mathbb{Z}^+}$ on a finite state-space S with transition matrix P. A function $f : S \to \mathbb{R}$ is considered as a column vector so that Pf makes sense as matrix multiplication.

   (a) Let $\mathbb{E}^x$ indicate that $X_0 = x \in S$. Check that

$$[Pf](x) = \mathbb{E}^x f(X_1).$$

   (b) Consider an arbitrary function $g : S \to \mathbb{R}$. Show that $g(X_n)$ is $\sigma(X_n)$-measurable. Set $\mathcal{F}_n = \sigma(X_k : 0 \leq k \leq n)$. Why is $g(X_n)$ is $\mathcal{F}_n$-measurable?

   (c) Fix $f : S \to \mathbb{R}$ and define

$$M_n = f(X_n) - f(X_0) - \sum_{k=0}^{n-1} [(P - I) f](X_k).$$

   Show that $(M_n, \mathcal{F}_n)$ is a martingale.

   (d) Call a function $f : S \to \mathbb{R}$ *sub-harmonic* if $f(x) \leq [Pf](x)$ for all x. Show that $f(X_n)$ is a sub-martingale w.r.t. the filtration $(\mathcal{F}_n)$. (This explains the "sub" in the definition of sub-martingale.)

3. (a) Given a σ-algebra $\mathcal{F}$, show that $A \in \mathcal{F}$ if and only 1_A is $\mathcal{F}$-measurable.

   (b) Let τ be a stopping time with respect to the filtration $(\mathcal{F}_t)_{t\in\mathbb{Z}^+}$. Show that $1_{\{\tau\geq t+1\}}$ is $\mathcal{F}_t$-measurable for all $t \in \mathbb{Z}^+$.

   (c) Let $(M_t, \mathcal{F}_t)_{t\in\mathbb{Z}^+}$ be a martingale and τ a stopping time w.r.t. the filtration $(\mathcal{F}_t)_{t\in\mathbb{Z}^+}$. Show directly (without appealing to martingale transforms) that the *stopped martingale*

$$M_t^\tau := M_{t\wedge\tau}$$

   is still a martingale w.r.t. the filtration $(\mathcal{F}_t)_{t\in\mathbb{Z}^+}$.

4. (a) Suppose that $X_1, X_2, ...$ are IID random variables with values in $\{-1, +1\}$ so that $\mathbb{P}(X_1 = 1) = p$, $\mathbb{P}(X_1 = -1) = 1 - p =: q$ for some fixed $p \in (0, 1)$. The process

$$S_0 = 0,\ S_t := X_1 + ... + X_t, \quad t \in \mathbb{N}.$$

is known as the *p/q-simple random walk* started at $0 \in \mathbb{R}$. Show that the processes

$$M_t := (q/p)^{S_t},\ \ N_t := S_t - \mathbb{E}S_t$$

are martingales w.r.t. the filtration given by $\mathcal{F}_t := \sigma(X_1, ..., X_t)$.

(b) Now assume $p = q = 1/2$, this gives rise to the *simple symmetric random walk* S. Assume the random walk is started $S_0 = 1$. Show that $\tau = \inf\{t \in \mathbb{Z}^+ : S_t = 0\}$ is a stopping time, possibly of value $+\infty$.

(c) Apply the martingale convergence theorem to see that the stopped martingale S^τ converges almost surely (to what?). Conclude that $\tau < \infty$ a.s..

$*$ (*d*) Show that the martingale S^τ does not converge in L^1.

(e) Now let S be the simple symmetric random walk S started at $S_0 = 0$. Let $a, b \in \mathbb{N}$, check that $\tau = \inf\{t \in \mathbb{Z}^+ : S_t = -a \text{ or } S_t = b\}$ is a stopping time. Why is $\tau < \infty$ almost surely?

(f) Use the optional stopping (sampling) theorem to compute the probability that S hits a before b. Find the expected time to hit $-a$ or b.

5. Let $X_1, X_2, \ldots$ be IID random variables with $\mathbb{E}X_1 = \mu$, $\mathbb{V}ar X_1 = \sigma^2$, and $\varphi(\theta) = \mathbb{E}e^{\theta X_1}$, finite-valued for all θ. Let $S_n = X_1 + \ldots + X_n$, $S_0 = 0$. Assuming that at time n the values of $X_1, \ldots, X_n$ are known, show that the following are martingales:

(a) $\{S_k^2 - \sigma^2 k : k \geq 0\}$ if and only if $\mu = 0$;

(b) $\{e^{\theta S_k} \varphi(\theta)^{-k} : k \geq 0\}$.

6. At time 1 an urn contains a white and a red ball. Take out a ball at random and replace it by two balls of the same color; this gives the new content of the urn at time 2. Keep iterating this procedure.

Let Y_n be the number of white balls in the urn at time n, and let $X_n = Y_n/(n + 1)$. Show that X_n is a.s. convergent to a random variable U. Compute the mean of U. Can you compute the variance of U?

7. Prove that the existence of an EMM implies no-arbitrage in the discrete multi-period setting. (You may assume $S_t^0 \equiv 1$ for all t.)

8. Suppose that over two periods a stock price moces on a binomial tree

$$
\begin{array}{ccccc}
 & & & & 45 \\
 & & & \diagup & \\
 & & 30 & - & 36 \\
 & \diagup & & & \\
15 & & & & \\
 & \diagdown & & & \\
 & & 12 & - & 16 \\
 & & & \diagdown & \\
 & & & & 10
\end{array}
$$

Determine for what values of the riskless rate r there is no arbitrage. If $r = 1/4$, determine the equivalent martingale measure. With this value of r, find the time-zero price and replicating portfolio for a European put option with strike 15 and expiry 2.

9. Consider a single-period trinomial model, with two assets, a riskless bond S^0 and a risky share S^1. We have initially that both are worth 1:

$$S_0^0 = 1 = S_0^1,$$

and that $S_1^0 = 1 + r$. The risky asset at time 1 will be worth a if the period was bad, b if the period was indifferent, and c if the period was good, $a < b < c$, and these are the only possibilities. We assume $a < 1 + r < c$.

   (a) Find all the equivalent martingale measures for this model.

   (b) For simplicity only, assume $r = 0$. Characterise all contingent claim with payoff $Y = (y_a, y_b, y_b)$ at time 1 that can be *replicated*, that is for which there exists $\theta \in \mathbb{R}^2$ such that

   $$Y = \theta \cdot S_1$$

   Determine the price of this contigent claim at time 0. Compute the expectation of Y with respect to any equivalent martingale measure. Conclusion?

   (c) How would your analysis extend to a single-period model with n assets?

10. A utility-maximising investor in a one-period binomial model has initial wealth $w_0 > 0$ and utility $U(x) = \sqrt{x}$. If S^0 denotes the riskless asset (worth 1 at time 0 and $1 + r$ at time 1), and S^1 denotes the risky asset, find the agent's optimal investment in the stock, and verify that this has the same sign as $\mathbb{E}[S_1^1 - (1 + r)S_0^1]$.