STOCHASTIC FINANCIAL MODELS: Examples 1 (of 4)

1 . Some useful facts about the Gaussian distribution for later reference. Here, Φ denotes the standard Gaussian cumulative distribution function.

(i) Suppose that $X \sim N(\mu, \sigma^2)$. Show that

(a) $\mathbb{E}[e^{\theta X} f(X)] = e^{\mu\theta + \sigma^2\theta^2/2}\mathbb{E}[f(X + \theta\sigma^2)]$ for all real θ and suitable f;

(b) for any nice function f, $\mathbb{E}[f(X)(X - \mu)] = \sigma^2 \mathbb{E}[f'(X)]$;

(c) $\mathbb{E}\Phi(\alpha X + \beta) = \Phi((\alpha\mu + \beta)/\sqrt{1 + \alpha^2\sigma^2})$ for all real α and β.

(ii) Suppose that (X, Y) has a jointly Gaussian distribution. Show that

(a) $\mathbb{E}[Y - \mathbb{E}Y|X] = \mathbb{C}\text{ov}(X, Y)(X - \mathbb{E}X)/\ \mathbb{V}\text{ar}(X)$;

(b) for any nice function f, $\mathbb{C}\text{ov}(f(X), Y) = \mathbb{E}[f'(X)]\ \mathbb{C}\text{ov}(X, Y)$.

2 . Prove that a function $U : \mathbb{R} \to (-\infty, \infty)$ is concave if and only if for all points $x_1 < y_1 \leq x_2 < y_2$, the inequality

$$\frac{U(y_1) - U(x_1)}{y_1 - x_1} \geq \frac{U(y_2) - U(x_2)}{y_2 - x_2}$$

holds. Deduce that for C^2 functions U, concavity is equivalent to the condition $U'' \leq 0$.

3 . Suppose that $U : \mathbb{R} \to \mathbb{R}$ is concave, and that the random variable ε has zero mean. Assuming that the function $\varphi : \mathbb{R} \to \mathbb{R}$ defined by

$$\varphi(\lambda) \equiv \mathbb{E}U(\mu + \lambda\varepsilon)$$

is everywhere finite-valued, prove that φ is concave. Prove also that φ is decreasing in $[0, \infty)$, and increasing in $(-\infty, 0]$.

4 . Suppose that U is a (nice) concave increasing function, and that $X \sim N(\mu, \sigma^2)$. Writing $\psi(\mu, \sigma) = \mathbb{E}U(X)$, show that ψ is concave in (μ, σ): for $0 \leq p \equiv 1 - q \leq 1$

$$\psi(p\mu_1 + q\mu_2, p\sigma_1 + q\sigma_2) \geq p\psi(\mu_1, \sigma_1) + q\psi(\mu_2, \sigma_2).$$

Show that ψ increases with μ and decreases with $\sigma \geq 0$, and hence show in the context of mean-variance analysis that, if all returns are jointly Gaussian, an investor maximising the expected utility of his final wealth will choose a mean-variance-efficient optimal portfolio.

5 . Suppose that an agent has expected-utility preferences over contingent claims, represented by a concave function U. Suppose further that if two contingent claims have the same mean and the same variance then he will be indifferent between them. By considering suitable distributions concentrated on three points, or otherwise, prove that the function U must be quadratic.

6 . Suppose an agent has 'utility' function

$$U(x) = x - \frac{1}{2}\varepsilon x^2$$

and may invest in a single stock, worth 1 at time 0 and some random amount S_1 at time 1, and in a bond, worth 1 at times 0 and 1. Given initial wealth w, show that his optimisation problem

$$\max_{\theta} EU(\theta S_1 + (w - \theta) + Y)$$

is solved by taking

$$\theta = \frac{E[(S_1 - 1)(1 - \varepsilon(w + Y))]}{\varepsilon E[(S_1 - 1)^2]},$$

where Y is a contingent claim to be received at time 1. Show that his maximised objective is then equal to

$$EU(w + Y) + \frac{1}{2}\frac{\left[E(S_1 - 1)(1 - \varepsilon(w + Y))\right]^2}{\varepsilon E[(S_1 - 1)^2]}.$$

In the special case where $Y \equiv 0$, show that this reduces to

$$U(w) + \frac{1}{2}(1 - \varepsilon w)^2 \frac{(ES_1 - 1)^2}{\varepsilon E[(S_1 - 1)^2]}.$$

Explain briefly how this would allow you to compute reservation bid prices for non-marketed contingent claims Z.

In next three questions, we consider a single-period model with d risky assets, worth $S_0 \equiv (S_0^1, \ldots, S_0^d)^T$ at time 0 and worth $S_1 \sim N(\mu, V)$ at time 1. The covariance matrix V is assumed non-singular. A riskless asset (asset 0) may be added to this market; the notations $\bar{S}_n \equiv (S_n^0, S_n^1, \ldots, S_n^d)^T$, $\bar{V}$, etc are then used.

7 . (a) An agent aims to maximise his expected utility of wealth at time 1:

$$\max_{\theta} \mathbb{E}U(\theta \cdot S_1),$$

subject to his budget constraint $\theta \cdot S_0 = w_0$. Show that (provided V is non-singular) the form of his optimal portfolio is

$$\theta^* = \gamma^{-1} V^{-1} \mu + \frac{\gamma w_0 - S_0 \cdot V^{-1}\mu}{\gamma S_0 \cdot V^{-1} S_0} \, V^{-1} S_0,$$

exactly as it would be for the case of CARA utility with coefficient of absolute risk aversion γ defined by

$$\gamma = -\mathbb{E}U''(\theta^* \cdot S_1)/\mathbb{E}U'(\theta^* \cdot S_1).$$

[HINT: Question 1 will help.]

(b) Suppose we now add the riskless asset. Show that with the same interpretation of γ, the investor's optimal portfolio is again exactly as it would be were his utility CARA with coefficient of absolute risk aversion γ.

8 . We have usually thought of the time-0 prices S_0 as given, and computed agents' optimal demands for the assets based on this. However, in an *equilibrium analysis*, the prices S_0 would be adjusted to *clear markets*. Suppose there is unit net supply of asset i, $i = 1, \ldots, d$, and zero net supply of asset 0; the (equilibrium) prices must be such that the total demand of all agents for each risky asset is 1, and for asset 0 is 0. Suppose that there are K agents in the market, agent k having CARA utility with coefficient of absolute risk aversion γ_k, and that agent k enters the market with a portfolio α_k of shares, $\sum_k \alpha_k = \mathbf{1}$. Without loss of generality, suppose $S_0^0 = 1$. Show that the market-clearing time-0 prices for the risky assets must be

$$S_0 = (\mu - \Gamma V \mathbf{1})/\mu^0,$$

where $\Gamma^{-1} = \sum_k \gamma_k^{-1}$.

9 . For simplicity, suppose that $\mu^0 = 1$. Suppose now that the agents decide to open a market in further assets $Y \equiv (Y^1, \ldots, Y^m)$, which are in zero net supply, and have the property that $(S_1^T, Y_1^T)^T$ is multivariate Gaussian, with mean $\bar{\mu}$ and covariance $\bar{V}$. Prove that *for this model* the time-0 prices of the assets S are unaffected, and that at time 0 none of the agents hold any of the assets Y. Find the time-0 prices of the assets Y.

10 . In a single-period investment model, risky assets S^1 and S^2 have moments $ES_1^1 = 3$, $ES_1^2 = 4$, $\mathbb{V}\mathrm{ar}(S_1^1) = 2$, $\mathbb{V}\mathrm{ar}(S_1^2) = 3$, and $\mathbb{C}\mathrm{ov}(S_1^1, S_1^2) = 2$. Find the portfolio which has minimum variance subject to having mean return m. What is the variance of this portfolio?