Part II Differential Geometry: Example Sheet 3 of 4

1. Let $\alpha : I \to S$ be a geodesic. Show that if α is a plane curve and $\ddot{\alpha}(t) \neq 0$ for some $t \in I$, then $\dot{\alpha}(t)$ is an eigenvector of the differential of the Gauss map at $\alpha(t)$. [*Hint: compare the normal n of α with the normal N of S.*]

2. Show that if all geodesics of a connected surface are plane curves, then the surface is contained in a plane or a sphere [*Hint: use the previous problem and Example sheet 2*].

3. Let $f : S_1 \to S_2$ be an isometry between two surfaces.

   (i) Let $\alpha : I \to S_1$ be a curve and V a vector field along α. Let $\gamma := f \circ \alpha$, and $W(t) := Df_{\alpha(t)}(V(t))$ the corresponding vector field along γ. Show that $DW/dt = Df_{\alpha(t)}(DV/dt)$, and hence that V parallel along α implies that W is parallel along γ.

   (ii) Deduce that f maps geodesics to geodesics.

4. Show that the equations for geodesics on a smooth surface may be written locally in terms of coordinates $(u(t), v(t))$ as

$$\frac{d}{dt}(E\dot{u} + F\dot{v}) = \frac{1}{2}(E_u\dot{u}^2 + 2F_u\dot{u}\dot{v} + G_u\dot{v}^2)$$
$$\frac{d}{dt}(F\dot{u} + G\dot{v}) = \frac{1}{2}(E_v\dot{u}^2 + 2F_v\dot{u}\dot{v} + G_v\dot{v}^2).$$

5. Show that there are no compact minimal surfaces in $\mathbb{R}^3$.

6. Let S be a regular surface without umbilical points. Prove that S is a minimal surface if and only if the Gauss map $N : S \to S^2$ satisfies

$$\langle DN_p(v_1), DN_p(v_2)\rangle = \lambda(p)\langle v_1, v_2\rangle$$

   for all $p \in S$ and all $v_1, v_2 \in T_pS$, where $\lambda(p) \neq 0$ is a number which depends only on p. By considering stereographic projection deduce that isothermal coordinates exist around a non planar point in a minimal surface.

7. Let $D \subset \mathbb{C}$, and f and g functions on D giving a Weierstrass representation of a parametrisation ϕ. Show that ϕ is an immersion if and only f vanishes only at the poles of g and the order of its zero at such a point is *exactly* twice the order of the pole of g.

8. Find D, f and g giving a Weierstrass representation of the catenoid, resp. the helicoid, with $\phi(u, v) = (a \cosh v \cos u, a \cosh v \sin u, av)$, resp. $\phi(u, v) = (a \sinh v \cos u, a \sinh v \sin u, au)$.

9. Show that the Gaussian curvature of the minimal surface determined by the Weierstrass representation is given by

$$K = -\left(\frac{4\,|g'|}{|f|\,(1 + |g|^2)^2}\right)^2.$$

   Show that either $K \equiv 0$ or its zeros are isolated. [*There is a way of doing this problem almost without calculations. Think about the relation between g and the Gauss map and the fact that stereographic projection is conformal.*]

10. The Weierstrass representation is not unique: if $\phi_{(f,g)} : D \to \mathbb{R}^3$ is the associated parametrization and $\alpha : W \to D$ is a bijective holomorphic map, then $\phi_{(f,g)} \circ \alpha$ is another representation of the same minimal surface and it must have the same form with different f and g (which should be specified). By choosing $\alpha(z) = g^{-1}(z)$, show that, locally around regular points of g at which g' is non-zero, we can assume that our pair (f, g) is of the form (F, id), for some local holomorphic function F. We denote such a representation by ϕ_F. Show that the minimal surfaces given by $\phi_{e^{-i\theta}F}$ for θ real are all locally isometric.

11. Let S_1 and S_2 be surfaces with Gauss curvature K_{S_1}, K_{S_2} respectively.

    (i) Suppose $f : S_1 \to S_2$ is a diffeomorphism with $K_{S_2}(f(x)) = K_{S_1}(x)$. Must f be an isometry?

    (ii) Suppose that for any geodesic $\gamma : I \to S_1$ parametrised by arc-length, $f \circ \gamma : I \to S_2$ is also a geodesic parametrised by arc-length. Must f be an isometry?

    (iii) Let $p_1 \in S_1$, $p_2 \in S_2$. Suppose S_1 and S_2 are locally isometric around p_1 and p_2. Show that there exists a geodesic $\gamma_1(t)$ emanating from p_1 and $\gamma_2(t)$ emanating from p_2 such that $K_{S_1}(\gamma_1(t)) = K_{S_2}(\gamma(t))$, and such that if $\alpha_i(t)$ is a geodesic emanating from p_i and making an angle θ from γ_i, for $i = 1, 2$ and fixed θ, then $K_{S_1}(\alpha_1(t)) = K_{S_2}(\alpha_2(t))$. Here all geodesics are parametrised by arc-length.

    (iv) Show conversely that if the above property is satisfied, then the S_i are locally isometric near the p_i. Deduce that if S_1 and S_2 have constant curvature $K_{S_1} = K_{S_2}$, then S_1 and S_2 are locally isometric. [*Hint: Consider geodesic polar coordinates and solve an ODE for G.*]

These questions are not part of the examples sheet. You should feel completely free to prioritise other things.

12. The intrinsic distance of a smooth embedded surface $S \subset \mathbb{R}^3$ is defined as follows. Given p and q in S let $d(p, q) = \inf_{\alpha \in \Omega(p,q)} \ell(\alpha)$. Show that d is a metric, which is compatible with the topology of S. If S is complete (and without boundary) the Hopf-Rinow theorem asserts that given two points p and q there exists a geodesic γ joining the points such that $d(p, q) = \ell(\gamma)$ and geodesics are defined for all $t \in \mathbb{R}$.

    (i) Show that if $f : S_1 \to S_2$ is an isometry, then $d_2(f(p), f(q)) = d_1(p, q)$ for all p and q in S_1.

    (ii) A geodesic $\gamma : [0, \infty) \to S$ is called a *ray leaving from* p if it realizes the distance between $\gamma(0)$ and $\gamma(s)$ for all $s \in [0, \infty)$. Let p be a point in a complete, noncompact surface S. Prove that S contains a ray leaving from p. [*You may assume that geodesics vary smoothly (hence continuously) with their initial conditions.*]

    (ii) Let S be a connected surface and let p be such that all geodesics through p are closed, i.e. all geodesics through p extend to smooth maps $\gamma : S^1 \to S$. Show that S is compact.

13. Show that any geodesic of the paraboloid of revolution $z = x^2 + y^2$ which is not a meridian intersects itself an infinite number of times [*Hint: use Clairaut's relation from IB. You may assume that no geodesic of a surface of revolution can be asymptotic to a parallel which is not itself a geodesic.*]

14. Take a good look at pictures of minimal surfaces online! You can get a free Mathematica license from the University if you want to play around with them yourself. Instructions (and the Mathematica code for the ones in class) are on Moodle. Or get some wire and make some soap films...