Part II Differential Geometry: Example Sheet 2 of 4

1. (i) Let $\alpha : I \to \mathbb{R}^3$ be a curve parametrized by arc length with curvature $k(s) \neq 0$ for all $s \in I$. Show that the torsion τ of α is given by
$$\tau(s) = -\frac{\langle \dot\alpha \wedge \ddot\alpha, \alpha^{(3)} \rangle}{|k(s)|^2}.$$
where $\alpha^{(3)}$ denotes the triple derivative with respect to s.

   (ii) Give an example of two curves $\alpha : [0,1] \to \mathbb{R}^3$, $\tilde\alpha : [0,1] \to \mathbb{R}^3$, parametrised by arc-length, such that $k(s) = \tilde k(s)$, and $\tau(s) = \tilde\tau(s)$ whenever $k(s) \neq 0$, but such that α and $\tilde\alpha$ are not related via Euclidean motion.

2. Let $\alpha : I \to \mathbb{R}^3$ be a curve parametrized by arc length with $\tau(s) \neq 0$ and $\dot k(s) \neq 0$ for all $s \in I$. Show that a necessary and sufficient condition for $\alpha(I)$ to lie on a sphere is that
$$R^2 + (\dot R)^2 T^2$$
is constant, where $R = 1/k$ and $T = 1/\tau$. [*To prove that the condition is necessary you need to differentiate three times* $|\alpha(s)|^2$. *To prove sufficiency, differentiate* $\alpha + Rn - \dot R T b$.]

3. Consider a closed plane curve inside a disk of radius r. Prove that there exists a point on the curve at which the curvature has absolute value $\geq 1/r$.

4. Let $\overline{AB}$ be a segment of straight line in the plane with endpoints A and B and let ℓ be a fixed number strictly bigger than the length of $\overline{AB}$. We consider curves joining A and B with length ℓ which lie on one side of the line through A and B; show that the curve which together with $\overline{AB}$ bounds the largest possible area is an arc of a circle passing through A and B. [*You may suppose that the isoperimetric inequality holds for piecewise smooth boundaries.*]

5. Let $\alpha : [0,\ell] \to \mathbb{R}^3$ be a curve parametrized by arc length with non-zero curvature everywhere. Suppose α has no self intersections, $\alpha(0) = \alpha(\ell)$ and it induces a smooth map from S^1 to $\mathbb{R}^3$ (i.e. α is a smooth simple closed curve). Let r be a positive number and consider the map $\phi : [0,\ell] \times [0,2\pi] \to \mathbb{R}^3$ given by:
$$\phi(s,v) = \alpha(s) + r(n(s)\cos v + b(s)\sin v)$$
where $n = n(s)$ and $b = b(s)$ are the normal and binormal vectors of α. The image T of ϕ is called the *tube* of radius r around α. It can be shown that for r sufficiently small T is an embedded surface. Prove that the area of T is $2\pi r\ell$.

6. (i) Let S be a surface that can be covered by connected coordinate neighbourhoods V_1 and V_2. Assume that $V_1 \cap V_2$ has two connected components W_1 and W_2, and that the Jacobian of the change of coordinates is positive on W_1 and negative on W_2. Prove that S is not orientable.

   (ii) Let $\phi : [0,2\pi] \times (-1,1) \to \mathbb{R}^3$ given by:
$$\phi(u,v) = \left((2 - v\sin(u/2))\sin u, (2 - v\sin(u/2))\cos u, v\cos(u/2)\right).$$
The image of ϕ is the Möbius strip. By considering the parametrizations given by ϕ restricted to $(0,2\pi) \times (-1,1)$ and
$$\psi(\bar u,\bar v) = \left((2 - \bar v\sin(\pi/4 + \bar u/2))\cos\bar u, -(2 - \bar v\sin(\pi/4 + \bar u/2))\sin\bar u, \bar v\cos(\pi/4 + \bar u/2)\right),$$
$(\bar u,\bar v) \in (0,2\pi) \times (-1,1)$, show that the Möbius strip is not orientable.

7. Show that the mean curvature H at $p \in S$ is given by
$$H = \frac{1}{\pi}\int_0^\pi k_n(\theta)\, d\theta,$$
where $k_n(\theta)$ is the normal curvature at p along a direction making an angle θ with a fixed direction.

8. Consider a surface of revolution parametrized by $\phi : (0, 2\pi) \times (a, b) \to \mathbb{R}^3$, where

$$\phi(u, v) = (f(v)\cos u, f(v)\sin u, g(v)).$$

Suppose f never vanishes and that the rotating curve is parametrized by arc-length, that is, $(f')^2+(g')^2 = 1$. Compute the Gaussian curvature and the mean curvature.

9. Let S be a compact orientable surface in $\mathbb{R}^3$. Show that the Gauss map is surjective and that it hits almost every unit vector the same number of times modulo 2. [*You may use the Jordan–Brouwer separation theorem.*] Show that S always has an elliptic point.

10. Show that if S is a connected surface in $\mathbb{R}^3$ such that every point is umbilic, then S is contained in a plane or a sphere. [Hint: Use that in a parametrization $\phi(u, v)$, $N_{uv} = N_{vu}$.]

These questions are not part of the examples sheet. They're different from typical 'starred' questions in other courses: they guide you through discovering further topics related to the course. They're not necessarily harder than the previous questions, but they're long, and you should feel completely free to prioritise other things.

11. Let p a point of a surface S such that the Gaussian curvature $K(p) \neq 0$ and let V be a small connected neighbourhood of p where K does not change sign. Define the *spherical area* $A_N(B)$ of a domain B contained in V as the area of $N(B)$ if $K(p) > 0$ or as minus the area of $N(B)$ if $K(p) < 0$ (N is the Gauss map). Show that

$$K(p) = \lim_{A\to 0} \frac{A_N(B)}{A(B)}$$

where $A(B)$ is the area of B and the limit is taken through a sequence of domains B_n that converge to p in the sense that any sphere around p contains all B_n for all n sufficiently large.

(This was the way Gauss introduced K.)

12. Let S be a surface with orientation N. Let $V \subset S$ be an open set and let $f : V \to \mathbb{R}$ be a nowhere vanishing smooth function. Let v_1 and v_2 be two smooth tangent vector fields in V such that at each point of V, v_1 and v_2 are orthonormal and $v_1 \wedge v_2 = N$.

(i) Prove that the Gaussian curvature K of V is given by

$$K = \frac{\langle D(fN)(v_1) \wedge D(fN)(v_2), fN\rangle}{f^3}.$$

(ii) Let f be the restriction of

$$\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}$$

to the ellipsoid E

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Show that the Gaussian curvature of E is

$$K = \frac{1}{a^2b^2c^2f^4}.$$