Part II Differential Geometry: Example Sheet 1 of 4

1. If X and Y are manifolds, show that $X \times Y$ is a manifold with $\dim X \times Y = \dim X + \dim Y$.

2. Let B_r be the open ball $\{x \in \mathbb{R}^k : |x| < r\}$. Show that the map
$$x \mapsto \frac{r\,x}{\sqrt{r^2 - |x|^2}}$$
is a diffeomorphism of B_r onto $\mathbb{R}^k$. (This implies that local parametrizations can always be chosen with all $\mathbb{R}^k$ as domain.)

3. If U is an open subset of $\mathbb{R}^n$ and V an open subset of $\mathbb{R}^m$ with $n \neq m$, prove that U and V are not diffeomorphic.

4. (i) Is the union of two coordinates axes in $\mathbb{R}^2$ a manifold?

   (ii) Prove that the hyperboloid in $\mathbb{R}^3$ given by $x^2 + y^2 - z^2 = a$ is a manifold for $a > 0$. What happens for $a = 0$? Find the tangent space at the point $(\sqrt{a}, 0, 0)$.

   (iii) Show that the solid hyperboloid $x^2 + y^2 - z^2 \leq a$ is a manifold with boundary ($a > 0$).

5. Prove that the set of all 2×2 matrices of rank 1 is a 3-dimensional submanifold of $\mathbb{R}^4$.

6. (Local immersion theorem.) The *canonical immersion* is the standard inclusion of $\mathbb{R}^k$ into $\mathbb{R}^n$ for $k \leq n$, that is
$$(x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_k, 0, \ldots, 0).$$
Let f be an immersion, $y = f(x)$. Show that there exist local coordinates around x and y such that f in these coordinates is the canonical immersion.

7. The *canonical submersion* is the standard projection of $\mathbb{R}^k$ onto $\mathbb{R}^l$ for $k \geq l$, that is
$$(x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_l).$$
(i) Let f be a submersion, $y = f(x)$. Show that there exist local coordinates around x and y such that f in these coordinates is the canonical submersion.

   (ii) Show that submersions are open maps, i.e. they carry open sets to open sets.

   (iii) If X is compact and Y connected, show that every submersion is surjective.

   (iv) Are there submersions of compact manifolds into Euclidean spaces?

8. Let $f : X \to Y$ be a smooth map and $y \in Y$ a regular value of f. Show that the tangent space to $f^{-1}(y)$ at a point $x \in f^{-1}(y)$ is given by the kernel of $Df_x : T_xX \to T_yY$.

9. Let $f : X \to X$ be a smooth map. f is called a *Lefschetz map* if given any fixed point x of f, $Df_x : T_xX \to T_xX$ does not have 1 as an eigenvalue. Prove that if X is compact and f is Lefschetz, then f has only finitely many fixed points.

10. (i) Prove that the boundary of a manifold with boundary is a manifold without boundary.

    (ii) Show that the square $[0, 1] \times [0, 1]$ is not a manifold with boundary.

11. Prove the following theorem due to Frobenius: let A be an $n \times n$ matrix all of whose entries are nonnegative. Then A has a nonnegative real eigenvalue. [Hint: consider the set $\{(x_1, \ldots, x_n) \in S^{n-1} : x_i \geq 0\ \forall i\}$ and apply the (topological) Brouwer fixed point theorem.]

12. A manifold is said to be *contractible* if the identity map is homotopic to a constant map. Show that a compact manifold without boundary is contractible only if it is the one-point space.

13. Let X be a compact manifold without boundary and Y a connected manifold with the same dimension as X.

   (i) Suppose that $f : X \to Y$ has $\deg_2(f) \neq 0$. Prove that f is onto.

   (ii) If Y is not compact, prove that $\deg_2(f) = 0$ for all smooth maps $f : X \to Y$.

These questions are not part of the examples sheet. They're different from typical 'starred' questions in other courses: they guide you through discovering further topics related to the course. They're not necessarily harder than the previous questions, but they're long, and you should feel completely free to prioritise other things.

14. (Bump functions.) (i) Let $\lambda : \mathbb{R} \to \mathbb{R}$ be given by $\lambda(x) = e^{-1/x^2}$ for $x > 0$ and $\lambda(x) = 0$ for $x \leq 0$. You know from Analysis I that λ is smooth. Show that $\tau(x) = \lambda(x-a)\,\lambda(b-x)$ is a smooth function, positive on (a,b) and zero elsewhere ($a < b$).

   (ii) Show that
$$\varphi(x) := \frac{\int_{-\infty}^{x} \tau}{\int_{-\infty}^{\infty} \tau}$$
   is smooth, $\varphi(x) = 0$ for $x < a$, $\varphi(x) = 1$ for $x > b$ and $0 < \varphi(x) < 1$ for $x \in (a,b)$.

   (iii) Finally construct a smooth function on $\mathbb{R}^n$ that equals 1 on the ball of radius a, zero outside the ball of radius b, and is strictly in between at intermediate points (here $0 < a < b$).

   These functions are very useful for smooth glueings. As an illustration, suppose $f_0, f_1 : X \to Y$ are smooth homotopic maps. Show that there exists a smooth homotopy $\tilde{F} : X \times [0,1] \to Y$ such that $\tilde{F}(x,t) = f_0(x)$ for all $t \in [0, 1/4]$ and $\tilde{F}(x,t) = f_1(x)$ for all $t \in [3/4, 1]$. Conclude that smooth homotopy is an equivalence relation.

15. (Morse functions). Let X be a k-manifold and $f : X \to \mathbb{R}$ a smooth function. Recall that a critical point is a point x for which df_x is not surjective, i.e. $df_x = 0$. A critical point is said to be *non-degenerate* if in local coordinates around x, the Hessian matrix $\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)$ has non-vanishing determinant. If all the critical points are non-degenerate, f is said to be a *Morse function.*

   (i) Show that the condition $\det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right) \neq 0$ is independent of the choice of chart.

   (ii) Suppose now that X is an open subset of $\mathbb{R}^k$. Given $a \in \mathbb{R}^k$, set
$$f_a(x) = f(x) + \langle x, a \rangle,$$
   where $\langle x, a \rangle$ denotes the usual inner product in $\mathbb{R}^k$. Show that f_a is a Morse function for a dense set of values of a. [Hint: consider $\nabla f : X \to \mathbb{R}^k$.]

   With a bit more work one can show that the same result is true if X is now any manifold and not just an open set of Euclidean space. In other words a "generic" smooth function is Morse.

   (iii) Show that the determinant function on $M(n)$ is Morse if $n = 2$, but not if $n > 2$.