Part II Differential Geometry: Example Sheet 4 of 4

1. Consider the standard (Euclidean) inner product on the space $M(n)$ of real $n \times n$ matrices, namely $\langle L, K\rangle = \mathrm{Tr}(LK^t)$ where K^t denotes the transpose matrix to K, and the induced metric on the tangent spaces to $X = \mathrm{O}(n) \subset M(n)$.

   For $A \in T_I X$, consider the curve $\alpha : \mathbb{R} \to M(n)$ given by $\alpha(t) = \exp(tA)$, as defined in lectures. Prove that α is a curve on X and that it is geodesic, that is $\alpha''(t) = A^2\alpha(t)$ is orthogonal to $T_{\alpha(t)}X$ for all $t \in \mathbb{R}$.

2. Using geodesic polar coordinates, show that given $p \in S$ we can express the Gaussian curvature as

$$K(p) = \lim_{r\to 0} \frac{3(2\pi r - L)}{\pi r^3},$$

   where L is the length of the geodesic circle of radius r. [*Hint: Taylor expansion.*]

3. Prove that on a surface of constant Gaussian curvature, the geodesic circles have constant geodesic curvature. Suppose that on a surface S, we have a point P with the property that locally around P the Gaussian curvature is constant along each geodesic circle; show that the geodesic curvature is also constant along these geodesic circles. Find the geodesic curvature of a parallel of latitude on the 2-sphere.

4. Let S be a connected surface and $f, g : S \to S$ two isometries. Assume that there exists $p \in S$, such that $f(p) = g(p)$ and $Df_p = Dg_p$. Show that $f(q) = g(q)$ for all $q \in S$.

5. (Geodesics are local minimizers of length.) Let p be a point on a surface S. Show that there exists an open set V containing p such that if $\gamma : [0, 1] \to V$ is a geodesic with $\gamma(0) = p$ and $\gamma(1) = q$ and $\alpha : [0, 1] \to S$ is a regular curve joining p to q, then

$$\ell(\gamma) \leq \ell(\alpha)$$

   with equality if and only if α is a monotonic reparametrization of γ.

6. Let P be a point on an embedded surface $S \subset \mathbb{R}^3$; consider the orthogonal parametrization $\phi : (-\epsilon, \epsilon)^2 \to V \subset S$ of a neighbourhood of P as constructed in lectures, where the curve $\phi(0, v)$ is a geodesic of unit speed, and for any $v_0 \in (-\epsilon, \epsilon)$ the curve $\phi(u, v_0)$ is a geodesic of unit speed. We showed that the first fundamental form was then $du^2 + G(u, v)dv^2$ for some smooth function G. Prove that $G(u, v) = 1$ for all u, v if and only if the curves $\phi(u_0, v)$ are geodesics for all $u_0 \in (-\epsilon, \epsilon)$.

7. Let S be a compact connected orientable surface in $\mathbb{R}^3$ which is not homeomorphic to a sphere. Prove that there are points on S where the Gaussian curvature is positive, negative, and zero.

8. Let S be a compact oriented surface with positive Gaussian curvature and let $N : S \to S^2$ be the Gauss map. Let γ be a simple closed geodesic in S, and let A and B be the regions which have γ as a common boundary. Show that $N(A)$ and $N(B)$ have the same area.

9. (i) Let S be an orientable surface with Gaussian curvature $K \leq 0$. Show that two geodesics γ_1 and γ_2 which start from a point $p \in S$ will not meet again at a point q in such a way that the traces (i.e. images) of γ_1 and γ_2 form the boundary of a domain homeomorphic to a disk.

   (ii) Let S be a surface homeomorphic to a cylinder and with negative Gaussian curvature. Show that S has at most one simple closed geodesic. Does the result remain true if "negative" is replaced with "non-positive"?

10. Let $\phi : U \to S$ be an orthogonal parametrization around a point p. Let $\alpha : [0, \ell] \to \phi(U)$ be a smooth simple closed curve parametrized by arc-length enclosing a domain R. Fix a unit vector $w_0 \in T_{\alpha(0)}S$ and consider $W(t)$ the parallel transport of w_0 along α. Let $\psi(t)$ be a differentiable determination of the angle from ϕ_u to $W(t)$. Show that

$$\psi(\ell) - \psi(0) = \int_R K\, dA.$$

Let S be a connected surface. Use the above to show that if the parallel transport between any two points does not depend on the curve joining the points, then the Gaussian curvature of S is zero.

11. If $a > 0$, calculate the curvature and torsion of the smooth curve given by

$$\alpha(s) = (a\cos(s/c), a\sin(s/c), bs/c) \quad \text{where } c = \sqrt{a^2+b^2}.$$

Suppose now that $\alpha : [0, 2\pi] \to \mathbb{R}^3$ is a smooth simple closed curve parametrized by arc-length with curvature everywhere positive. If both k and τ are constant, show that $k = 1$ and $\tau = 0$. If k is constant and τ is not identically zero, show that $k > 1$. If α is knotted and τ is constant, show that $k(s) > 2$ for some $s \in [0, 2\pi]$.

These questions are not part of the examples sheet. They're different from typical 'starred' questions in other courses: they guide you through discovering further topics, and complete a circle of ideas in the course. Their content is certainly not examinable.

12. (The Poincaré-Hopf theorem.) Let S be an oriented surface and $V : S \to \mathbb{R}^3$ a smooth vector field, that is, $V(p) \in T_pS$ for all $p \in S$. We say that p is *singular* if $V(p) = 0$. A singular point p is *isolated* if there exists a neighbourhood of p in which V has no other zeros. The singular point p is *non-degenerate* if $DV_p : T_pS \to T_pS$ is a linear isomorphism (can you see why dV_p takes values in T_pS?). Show that if a singular point is non-degenerate, then it is isolated.

To each isolated singular point p we associate an integer called the *index* of the vector field at p as follows. Let $\phi : U \to S$ be an orthogonal parametrization around p compatible with the orientation. Let $\alpha : [0, l] \to \phi(U)$ be a regular piecewise smooth simple closed curve so that p is the only zero of V in the domain enclosed by α. Let $\varphi(t)$ be some differentiable determination of the angle from ϕ_u to $V(t) := V \circ \alpha(t)$. Since α is closed, there is an integer I (the index) defined by

$$2\pi\, I := \varphi(l) - \varphi(0).$$

(i) Show that I is independent of the choice of parametrization (Hint: use an ealier problem). One can also show that I is independent of the choice of curve α, but this is a little harder. Also one can prove that if p is non-degenerate, then $I = 1$ if DV_p preserves orientation and $I = -1$ if dV_p reverses orientation.

(ii) Draw some pictures of vector fields in $\mathbb{R}^2$ with an isolated singularity at the origin. Compute their indices.

(iii) Suppose now that S is compact and that V is a smooth vector field with isolated singularities. Consider a triangulation of S such that

- every triangle is contained in the image of some orthogonal parametrization;
- every triangle contains at most one singular point;
- the boundaries of the triangles contain no singular points and are positively oriented.

Show that

$$\sum_i I_i = \frac{1}{2\pi}\int_S K\, dA = \chi(S).$$

Thus, you have proved that the sum of the indices of a smooth vector field with isolated singularities on a compact surface is equal to the Euler characteristic (Poincaré–Hopf theorem). Conclude that a surface homeomorphic to S^2 cannot be combed.

Finally, suppose $f : S \to \mathbb{R}$ is a Morse function and consider the vector field given by the gradient of f, i.e., $\nabla f(p)$ is uniquely determined by $\langle \nabla f(p), v\rangle = Df_p(v)$ for all $v \in T_pS$. Use the Poincaré–Hopf theorem to show that $\chi(S)$ is the number of local maxima and minima minus the number of saddle points. Use this to find the Euler characteristic of a surface of genus two.

You can read more about this in Chap. VI of Milnor's 'Topology from the differential viewpoint'.

13. (The degree of the Gauss map.) Let S be a compact oriented surface and let $N : S \to S^2$ be the Gauss map. Consider $y \in S^2$ a regular value. Rather than counting their preimages modulo 2 as we did in the first lectures, we will count them with sign. Let $N^{-1}(y) = \{p_1, \dots, p_n\}$. Let $\varepsilon(p_i)$ be $+1$ if DN_{p_i} preserves orientation ($K(p_i) > 0$), and -1 if DN_{p_i} reverses orientation ($K(p_i) < 0$). Now let

$$\deg(N) := \sum_i \varepsilon(p_i).$$

As in the case of the degree mod 2, it can be shown that the sum on the right hand side is independent of the regular value and $\deg(N)$ turns out to be an invariant of the homotopy class of N.

Now, choose $y \in S^2$ such that y and $-y$ are regular values of N. Why can we do so? Let V be the vector field on S given by

$$V(p) := \langle y, N(p) \rangle N(p) - y.$$

(i) Show that the index of V at a zero p_i is $+1$ if DN_{p_i} preserves orientation and -1 if DN_{p_i} reverses orientation.

(ii) Show that the sum of the indices of V equals twice the degree of N.

(iii) Show that $\deg(N) = \chi(S)/2$.