DIFFERENTIAL GEOMETRY EXAMPLES 1

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Comments/corrections are welcome, and may be e-mailed to me at `pmhw@dpmms.cam.ac.uk`.

1. If X and Y are manifolds, show that $X \times Y$ is a manifold with $\dim X \times Y = \dim X + \dim Y$.

2. Let B_r be the open ball $\{x \in \mathbb{R}^k : \ |x| < r\}$. Show that the map

$$x \mapsto \frac{r\,x}{\sqrt{r^2 - |x|^2}}$$

is a diffeomorphism of B_r onto $\mathbb{R}^k$. (This implies that local parametrizations can always be chosen with all $\mathbb{R}^k$ as domain.)

3. If U is an open subset of $\mathbb{R}^n$ and V an open subset of $\mathbb{R}^m$ with $n \neq m$, prove that U and V are not diffeomorphic.

4. (i) Is the union of two coordinates axes in $\mathbb{R}^2$ a manifold?

(ii) Prove that the hyperboloid in $\mathbb{R}^3$ given by $x^2 + y^2 - z^2 = a$ is a manifold for $a > 0$. What happens for $a = 0$? Find the tangent space at the point $(\sqrt{a}, 0, 0)$.

(iii) Show that the solid hyperboloid $x^2 + y^2 - z^2 \leq a$ is a manifold with boundary $(a > 0)$.

5. Recall that a submersion is a smooth map $f : X \to Y$ such that df_x is surjective for all $x \in X$. The *canonical submersion* is the standard projection of $\mathbb{R}^k$ onto $\mathbb{R}^l$ for $k \geq l$, that is

$$(x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_l).$$

(i) Let f be a submersion, $y = f(x)$. Show that there exist local coordinates around x and y such that f in these coordinates is the canonical submersion.

(ii) Show that submersions are open maps, i.e. they carry open sets to open sets.

(iii) If X is compact and Y connected, show that every submersion is surjective.

(iv) Are there submersions of compact manifolds into Euclidean spaces?

6. (i) Let $f : X \to Y$ be a smooth map and y a regular value of f. Show that the tangent space to $f^{-1}(y)$ at a point x is given by the kernel of $df_x : T_xX \to T_yY$.

(ii) Show that the orthogonal group $O(n)$ is compact and that its tangent space at the identity is given by all matrices H for which $H^t = -H$.

7. Prove that the set of all 2×2 matrices of rank 1 is a 3-dimensional submanifold of $\mathbb{R}^4$.

8. For which values of a does the hyperboloid $x^2 + y^2 - z^2 = 1$ intersect the sphere $x^2 + y^2 + z^2 = a$ transversally? What does the intersection look like for different values of a?

9. Let $f : X \to X$ be a smooth map. f is called a *Lefschetz map* if given any fixed point x of f, $df_x : T_xX \to T_xX$ does not have 1 as an eigenvalue. Prove that if X is compact and f is Lefschetz, then f has only finitely many fixed points.

10. Prove the following theorem due to Frobenius: let A be an $n \times n$ matrix all of whose entries are nonnegative. Then A has a nonnegative real eigenvalue. [Hint: consider the set $\{(x_1, \ldots, x_n) \in S^{n-1} : \ x_i \geq 0 \ \forall i\}$ and apply the (topological) Brouwer fixed point theorem.]

11. A manifold is said to be *contractible* if the identity map is homotopic to a constant map. Show that a compact manifold without boundary is contractible only if it is the one-point space.

12. Let X be a compact manifold without boundary and Y a connected manifold with the same dimension as X.

(i) Suppose that $f : X \to Y$ has $\deg_2(f) \neq 0$. Prove that f is onto.

(ii) If Y is not compact, prove that $\deg_2(f) = 0$ for all smooth maps $f : X \to Y$.

13. (i) Prove that the boundary of a manifold with boundary is a manifold without boundary.

(ii) Show that the square $[0,1] \times [0,1]$ is not a manifold with boundary.

14. (i) Let $\lambda : \mathbb{R} \to \mathbb{R}$ be given by $\lambda(x) = e^{-1/x^2}$ for $x > 0$ and $\lambda(x) = 0$ for $x \leq 0$. You know from Analysis I that λ is smooth. Show that $\tau(x) = \lambda(x-a)\,\lambda(b-x)$ is a smooth function, positive on (a,b) and zero elsewhere ($a < b$).

(ii) Show that

$$\varphi(x) := \frac{\int_{-\infty}^{x} \tau}{\int_{-\infty}^{\infty} \tau}$$

is smooth, $\varphi(x) = 0$ for $x < a$, $\varphi(x) = 1$ for $x > b$ and $0 < \varphi(x) < 1$ for $x \in (a,b)$.

(iii) Finally construct a smooth function on $\mathbb{R}^n$ that equals 1 on the ball of radius a, zero outside the ball of radius b, and is strictly in between at intermediate points (here $0 < a < b$).

These functions are very useful for smooth glueings. As an illustration, suppose $f_0, f_1 : X \to Y$ are smooth homotopic maps. Show that there exists a smooth homotopy $\tilde{F} : X \times [0,1] \to Y$ such that $\tilde{F}(x,t) = f_0(x)$ for all $t \in [0, 1/4]$ and $\tilde{F}(x,t) = f_1(x)$ for all $t \in [3/4, 1]$. Conclude that smooth homotopy is an equivalence relation.

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15. Show that there is a smooth map $\mathbf{R} \to \mathbf{R}$ for which the critical values are precisely the rational numbers.

16 (Morse functions). Let X be a k-manifold and $f : X \to \mathbb{R}$ a smooth function. Recall that a critical point is a point x for which df_x is not surjective, i.e. $df_x = 0$. A critical point is said to be *non-degenerate* if in local coordinates around x, the Hessian matrix $\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)$ has non-vanishing determinant. If all the critical points are non-degenerate, f is said to be a *Morse function.*

(i) Show that the condition $\det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right) \neq 0$ is independent of the choice of chart.

(ii) Suppose now that X is an open subset of $\mathbb{R}^k$. Given $a \in \mathbb{R}^k$, set

$$f_a(x) = f(x) + \langle x, a \rangle,$$

where $\langle x, a \rangle$ denotes the usual inner product in $\mathbb{R}^k$. Show that f_a is a Morse function for a dense set of values of a. [Hint: consider $\nabla f : X \to \mathbb{R}^k$.]

With a bit more work one can show that the same result is true if X is now any manifold and not just an open set of Euclidean space. In other words a "generic" smooth function is Morse.

(iii) Show that the determinant function on $M(n)$ is Morse if $n = 2$, but not if $n > 2$.