Part IID DIFFERENTIAL GEOMETRY (Mich. 2011): Example Sheet 2

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1. Let $\alpha : I \to \mathbb{R}^3$ be a curve parametrized by arc length with curvature $k(s) \neq 0$ for all $s \in I$. Show that the torsion τ of α is given by

$$\tau(s) = -\frac{\langle \dot\alpha \times \ddot\alpha, \dddot\alpha \rangle}{|k(s)|^2}.$$

2. (i) Let $\alpha : I \to \mathbb{R}^3$ be a curve parametrized by arc length with $\tau(s) \neq 0$ and $\dot k(s) \neq 0$ for all $s \in I$. Show that a necessary and sufficient condition for $\alpha(I)$ to lie on a sphere is that

$$R^2 + (\dot R)^2 T^2$$

is constant, where $R = 1/k$ and $T = 1/\tau$. [To prove that the condition is necessary you need to differentiate three times $|\alpha(s)|^2$. To prove sufficiency, differentiate $\alpha + Rn - \dot R T b$.]

(ii) Show that if α is a closed smooth curve lying on a sphere, then there exists a point $\alpha(s_0)$ such that $\tau(s_0) = 0$. [Hint: differentiate the third derivative of $|\alpha(s)|^2$ obtained in (i).]

3. Consider a closed plane curve inside a disk of radius r. Prove that there exists a point on the curve at which the curvature has absolute value $\geq 1/r$.

4. Let $\overline{AB}$ be a segment of straight line in the plane with endpoints A and B and let ℓ be a fixed number strictly greater than the length of $\overline{AB}$. Show that the curve joining A and B with length ℓ and such that together with $\overline{AB}$ bounds the largest possible area is an arc of a circle passing through A and B.
[You may suppose that the isoperimetric inequality holds for piecewise smooth boundaries.]

5. Let $\varphi : U \to S$ be a parametrization of a surface S in R^3. Show that

$$|\varphi_u \times \varphi_v| = \sqrt{EG - F^2}.$$

6. Let $\alpha \colon [0, \ell] \to \mathbb{R}^3$ be a curve parametrized by arc length with non-zero curvature everywhere. Suppose α has no self intersections, $\alpha(0) = \alpha(\ell)$ and it induces a smooth map from S^1 to $\mathbb{R}^3$ (i.e. α is a smooth simple closed curve). Let r be a positive number and consider the map $\varphi : [0, \ell] \times [0, 2\pi] \to \mathbb{R}^3$ given by:

$$\varphi(s, v) = \alpha(s) + r\big(n(s)\cos v + b(s)\sin v\big),$$

where $n = n(s)$ and $b = b(s)$ are the normal and binormal vectors of α. The image T of φ is called the *tube* of radius r around α. It can be shown that for r sufficiently small T is a surface. Prove that the area of T is $2\pi r\ell$.

7. (i) Let S be a surface that can be covered by connected coordinate neighbourhoods V_1 and V_2. Assume that $V_1 \cap V_2$ has two connected components W_1 and W_2, and that the Jacobian of the change of coordinates is positive on W_1 and negative on W_2. Prove that S is not orientable.

(ii) Let $\varphi : [0, 2\pi] \times (-1, 1) \to \mathbb{R}^3$ given by

$$\varphi(u, v) = \big((2 - v\sin(u/2))\sin u,\ (2 - v\sin(u/2))\cos u,\ v\cos(u/2)\big).$$

The image of φ is the Möbius strip. By considering the parametrizations given by φ restricted to $(0, 2\pi) \times (-1, 1)$ and

$$\psi(\bar{u}, \bar{v}) = \big((2 - \bar{v}\sin(\pi/4 + \bar{u}/2))\cos\bar{u},\ -(2 - \bar{v}\sin(\pi/4 + \bar{u}/2))\sin\bar{u},\ \bar{v}\cos(\pi/4 + \bar{u}/2)\big),$$

$(\bar{u}, \bar{v}) \in (0, 2\pi) \times (-1, 1)$, show that the Möbius strip is not orientable.

8. Show that the mean curvature H at $p \in S$ is given by

$$H = \frac{1}{\pi} \int_0^{\pi} k_n(\theta)\, d\theta,$$

where $k_n(\theta)$ is the normal curvature at p along a direction making an angle θ with a fixed direction.

9. Consider a surface of revolution parametrized by $\varphi : (0, 2\pi) \times (a, b) \to \mathbb{R}^3$, where

$$\varphi(u, v) = \big(f(v)\cos u, f(v)\sin u, g(v)\big).$$

Suppose f never vanishes and that the rotating curve is parametrized by arc-length, that is, $(f')^2 + (g')^2 = 1$. Compute the Gaussian curvature and the mean curvature.

10. (i) Determine an equation for the *tractrix*, which is the curve such that the length of the segment of the tangent line between the point of tangency and some fixed line l in the plane—which does not meet the curve—is a constant equal to 1.

(ii) Rotate the tractrix about the line l to obtain a surface of revolution (called the *pseudosphere*). Compute its Gaussian curvature.

11. Let S be a compact orientable surface in $\mathbb{R}^3$. Show that the Gauss map is surjective and that it hits almost every direction the same number of times modulo 2. [You may use the Jordan–Brouwer separation theorem.] Show that S always has an elliptic point.

12. If φ is an orthogonal parametrization, i.e. $F = 0$, show that the Gauss formula yields:

$$K = -\frac{1}{2\sqrt{EG}} \left\{ \left(\frac{E_v}{\sqrt{EG}} \right)_v + \left(\frac{G_u}{\sqrt{EG}} \right)_u \right\}.$$

13. Let p a point of a surface S such that the Gaussian curvature $K(p) \neq 0$ and let V be a small connected neighbourhood of p where K does not change sign. Define the *spherical area* $A_N(B)$ of a domain B contained in V as the area of $N(B)$ if $K(p) > 0$ or as minus the area of $N(B)$ if $K(p) < 0$ (N is the Gauss map). Show that

$$K(p) = \lim_{A \to 0} \frac{A_N(B)}{A(B)},$$

where $A(B)$ is the area of B and the limit is taken through a sequence of domains B_n that converge to p in the sense that any sphere around p contains all B_n for all n sufficiently large.

(This was the way Gauss introduced K.)

14. Show that if S is a connected surface in $\mathbb{R}^3$ such that every point is umbilic, then S is part of a plane or a sphere. [Hint: use that in a parametrization $\varphi(u, v)$, $N_{uv} = N_{vu}$.]