MATHEMATICAL TRIPOS PART II (2021–2022)
CODING AND CRYPTOGRAPHY
EXAMPLE SHEET 1 OF 4

1 For a code $f : \Sigma_1 \to \Sigma_2^*$ and a code $f' : \Sigma_1' \to \Sigma_2'^*$ the product code is $g : \Sigma_1 \times \Sigma_1' \to (\Sigma_2 \cup \Sigma_2')^*$ given by $g(x, y) = f(x)f'(y)$. Show that the product of two prefix-free codes is prefix-free, but that the product of a decipherable code and a prefix-free code need not even be decipherable.

2 Jensen's inequality states that if $f : \mathbb{R} \to \mathbb{R}$ is a convex function and $p_1, \ldots, p_n$ is a probability distribution (*i.e.* $0 \leqslant p_i \leqslant 1$ and $\sum p_i = 1$) then $f(\sum p_i x_i) \leq \sum p_i f(x_i)$ for any $x_1, \ldots, x_n \in \mathbb{R}$. Deduce Gibbs' inequality from Jensen's inequality applied to the convex function $f(x) = -\log x$.

3 Show that $H(p_1, p_2, p_3) \leq H(p_1, 1 - p_1) + (1 - p_1)$ and determine when equality occurs.

4 Use the methods of Shannon-Fano and Huffman to construct prefix-free binary codes for messages $\mu_1, \ldots, \mu_5$ emitted (i) with equal probabilities, or (ii) with probabilities $0.3, 0.3, 0.2, 0.15, 0.05$. Compare the expected word lengths in each case.

5 Messages $\mu_1, \ldots, \mu_5$ are emitted with probabilities $0.4, 0.2, 0.2, 0.1, 0.1$. Determine whether there are optimal binary codings with (i) all but one codeword of the same length, or (ii) each codeword a different length.

6 A binary Huffman code is used for encoding symbols $1, \ldots, m$ occurring with probabilities $p_1 \geq p_2 \geq \cdots \geq p_m > 0$ where $\sum_{1 \leq j \leq m} p_j = 1$. Let s_1 be the length of the shortest codeword and s_m the length of the longest codeword. Determine the maximal and minimal values of s_1 and s_m and find binary trees for which they are attained.

7 Show that if an optimal binary code has word lengths $s_1, \ldots, s_m$ then

$$m \log m \leq s_1 + \cdots + s_m \leq (m^2 + m - 2)/2.$$

8 Consider 64 messages M_j with the following properties: M_1 has probability $1/2$, M_2 has probability $1/4$ and M_j has probability $1/248$ for $3 \leqslant j \leqslant 64$. Explain why, if we use (binary) codewords of equal length, then the length of the codeword must be at least 6. By using the ideas of Huffman's algorithm (you should not need to go through all the steps) obtain a set of codewords such that the *expected* length of a codeword sent is no more than 3.

9 Suppose that a SARS-CoV-2 infection is known to originate in exactly one of m rooms in College, the probability it originates in the j^{th} being p_j. A health inspector has samples from all of the m rooms and by testing the pooled samples from a set A of them can determine with certainty whether the infection originates in A or its complement. Let $N(p_1, \ldots, p_m)$ denote the minimum expected number of such tests needed to locate the infection. Show that $H(p_1, \ldots, p_m) \leq N(p_1, \ldots, p_m) < H(p_1, \ldots, p_m) + 1$, and determine when the lower bound is attained.

10 Extend the definition of entropy to a random variable taking values in the non-negative integers. Compute the expected value $E(X)$ and entropy $H(X)$ of a random variable X with $P(X = k) = p(1-p)^k$. Show that among non-negative integer valued random variables with the same expected value, X achieves the maximum possible entropy.

11 A source emits messages $\mu_1, \ldots, \mu_m$ with non-zero probabilities $p_1, \ldots, p_m$. Let S be the codeword length random variable for a decipherable code $f : \Sigma_1 \to \Sigma_2^*$ where $\Sigma_1 = \{\mu_1, \ldots, \mu_m\}$ and $|\Sigma_2| = a$. Show that the minimum possible value of $E(a^S)$ satisfies

$$\left(\sum_{i=1}^{m} \sqrt{p_i}\right)^2 \leq E(a^S) < a\left(\sum_{i=1}^{m} \sqrt{p_i}\right)^2.$$

(Hint: The Cauchy-Schwarz inequality.)

12 (i) In lectures we only described Huffman coding in the binary case, *i.e.* $a = 2$. In general we add extra messages of probability zero so that the number of messages m satisfies $m \equiv 1 \pmod{a-1}$. Then at each stage we group together the a smallest probabilities. Carry this out for a ternary coding of a source with probabilities $0.2, 0.2, 0.15, 0.15, 0.1, 0.1, 0.05, 0.05$.

(ii) Show that if a ternary decipherable code of size m meets the lower bound in the noiseless coding theorem then m is odd.

Further Problems

Note: the examples above are minimal to cover the course; you are encouraged to try those below also.

13 Into the Martin Handicap at Royal Basket are entered m horses, the probability that the i^{th} horse wins being p_i. The odds offered on each horse are a_i–for–1, *i.e.* a bet of x pounds on the i^{th} horse will yield $a_i x$ pounds if the horse wins, and nothing otherwise. A gambler bets a proportion b_i of his wealth on horse i, with $\sum_{i=1}^{m} b_i = 1$. He seeks to maximise $W = \sum_{i=1}^{m} p_i \log(a_i b_i)$. Solve to find the b_i that maximise W. Show that, in the case when all odds are the same, this maximum and the entropy $H(p_1, \ldots, p_m)$ sum to a constant.

14 A balance puzzle: you are given m apparently identical coins, one of which may be a forgery. Forged coins are either too light or too heavy. You are also given a balance, on which you may place any of the coins you like. The coins placed in either pan may be together heavier or lighter than those in the other pan or the pans may balance.

You are allowed at most 3 uses of the balance. Show that if $m > 13$ then you cannot be sure of detecting the forgery and its nature. [Optional] Show that for $m = 12$ three weighings suffice.

This problem 'is said to have been planted during the war ... by enemy agents since Operational Research spent so many man-hours on its solution.'[1]]

[1] The quotation is lifted from Dan Pedoe's *The Gentle Art of Mathematics* (Dover reprint, 1982) which also gives an attractive solution. Niobe, the protagonist of Piers Anthony's novel *With a Tangled Skein*, must solve the twelve-coin variation of this puzzle to find her son in Hell: Satan has disguised the son to look identical to eleven other demons, and he is heavier or lighter depending on whether he is cursed to lie or able to speak truthfully. In the episode 'Captain Peralta' of *Brooklyn Nine-Nine*, Holt presents to his team a version of the twelve-coin problem involving twelve men and a seesaw. The original 12 coin version was solved in 1945 by Grossman.

15 In an unreleased episode of *The Queen's Gambit*, there is a game on a standard chessboard in which one player (Beth) has to guess where her opponent has placed the Queen. Beth is allowed six questions which must be answered truthfully by a yes/no reply. Prove that there is a strategy by which Beth can always win this game, but that she cannot ensure winning if she is allowed only five questions.

In the forthcoming sequel, the game is played on an $n \times n$ chessboard. How many questions does Beth need in order to be certain of winning?

SM, Lent Term 2022
Comments on and corrections to this sheet may be emailed to `sm137@cam.ac.uk`