# PART II AUTOMATA AND FORMAL LANGUAGES
# MICHAELMAS 2025-26
# EXAMPLE SHEET 4

(1) Let $A$ be a recursive set, and define the set

$$B = \{2n \mid n \in A\} \cup \{2n + 1 \mid n \in \mathbb{K}\}.$$

(a) Is $B$ recursive? If not, which of $B$ and $\mathbb{N} \setminus B$ are r.e. (if any), and why?

(b) By replacing $A$ in the construction of $B$ with a suitably chosen set, construct a set $C \subseteq \mathbb{N}$ such that neither $C$ nor $\mathbb{N} \setminus C$ are r.e.

(2) Give examples, with proofs, of infinite collections of recursive sets whose union:

(a) Is recursive.

(b) Is r.e. but not recursive.

(c) Is not r.e., and its complement is not r.e. either.

(3) Give an example of an infinite collection of recursive sets $\{W_n\}_{n \in I}$, whose index set $I$ is r.e., for which

$$\bigcap_{n \in I} W_n$$

is not r.e.

(4) Let $\Sigma$ be a finite alphabet, and $L$ be a regular language over $\Sigma$.

(a) Fix an ordering of $\Sigma$, and use this to describe a bijection $b$ from $\Sigma^*$ to $\mathbb{N}$.

(b) Using this ordering of $\Sigma^*$, show that $\{n \in \mathbb{N} \mid b^{-1}(n) \in L\}$ is a recursive set.

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(5) Show that if $g : \mathbb{N}^2 \to \mathbb{N}$ is total recursive, then there is some $e \in \mathbb{N}$ such that

$$f_{e,1}(y) = g(e, y) \quad \forall y \in \mathbb{N}.$$

Use this to show there exists some $m \in \mathbb{N}$ such that $W_m$ has exactly $m$ elements.

(6) Prove that the set $\{n \in \mathbb{N} \mid |W_n| > 5\}$ is r.e., but not recursive.

(7) Show that there is an $n \in \mathbb{N}$ such that $W_n = \{n\}$.

(8) Assume that we have two subsets $X, Y$ of $\mathbb{N}$ which are neither $\mathbb{N}$ nor $\emptyset$. Show that the Turing join $X \oplus Y$ is the least upper bound of $X$ and $Y$ with respect to the relation $\leq_\mathrm{m}$, namely if we have $Z \subseteq \mathbb{N}$ with both $X, Y \leq_m Z$, then $X \oplus Y \leq_m Z$.

(9) Show that **Emp** $\leq_m \mathbb{N} \setminus \mathbb{K}$ and $N \setminus \mathbb{K} \leq_m$ **Emp**. [*Hint.* Recall that the set $\mathbb{N} \setminus \mathbb{K}$ is $\Pi_1$-complete. Why?]

(10) Prove that $\mathbb{K}$ is not an index set.

(11) Prove that **Inf** and **Tot** are neither $\Sigma_1$ nor $\Pi_1$.

(12) Given an explicit code $m$ for a register machine $P_m$, consider the set

$$T_m := \{n \in \mathbb{N} \,|W_m \subseteq W_n\}.$$

Construct two explicit codes $m, m'$ such that $T_m$ is recursive, and $T_{m'}$ is not r.e.

(13) Let $g : \mathbb{N}^k \to \mathbb{N}$ be a total computable function. Consider $\mathbf{Eq}(g) := \{n : f_{n,k} = g\}$ and show that **Tot** $\leq_m \mathbf{Eq}(g)$.