Automata & Formal Languages
Michaelmas Term 2024
Part II of the Mathematical Tripos
University of Cambridge
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Example Sheet #4

(42) In Example (31), you gave the concrete definition of a register machine M. Write down its code in $\mathbb{B}$ and calculate $\#(\mathrm{code}(M))$.

(43) Let φ be a total computable function. A word w is called a *fixed point of* φ if $f_{\varphi(w),1} = f_{w,1}$. The *Recursion Theorem* or *Fixed Point Theorem* states that every total computable function has a fixed point.

(a) Argue that there is a total computable function h such that

$$f_{h(u),1}(v) := \begin{cases} f_{f_{u,1}(u)}(v) & \text{if } u \in \mathbf{K} \text{ and} \\ \uparrow & \text{otherwise.} \end{cases}$$

(b) Prove the Recursion Theorem.
[*Hint.* Let e be such that $f_{e,1} = \varphi \circ h$ and $w := h(e)$, where h is as in (a).]

(44) Show that there is a w such that $\mathrm{W}_w = \{w\}$ and a w such that $|\mathrm{W}_w| = |w|$.

(45) Let $f : \mathbb{B}^2 \to \mathbb{B}$ be a partial computable function. Show that the following sets are computably enumerable:

(a) $\{w\,;$ there are three distinct words v such that $f(w,v)\downarrow\}$;
(b) $\{w\,;$ there is a word v of even length such that $f(w,v)\downarrow\}$;
(c) $\{w\,;$ there is a word v such that $f(w,v) = w * v\}$.

(46) Let $f : \mathbb{B} \to \mathbb{B}$ be a total bijective function. Show that f is computable if and only if f^{-1} is computable.

(47) Let $L \subseteq \mathbb{B}$ be non-empty. Show that L is computably enumerable if and only if there is a total computable function f such that $L = \mathrm{ran}(f)$.

(48) Suppose X is computably enumerable. Show that $\bigcup_{v\in X} \mathrm{W}_v$ is computably enumerable. Deduce that the class of computably enumerable sets is closed under finite unions.

(49) Show that there is a computably enumerable set X such that for all $w \in X$, W_w is a computable set, but $\bigcap_{w\in X} \mathrm{W}_w$ is not computably enumerable.

(50) Assume that $\leq$ is a partial preorder on X, i.e., reflexive and transitive, and define $\equiv$ by $x \equiv y$ if and only if $x \leq y$ and $y \leq x$. Show that $\equiv$ is an equivalence relation and that $\leq$ respects the equivalence classes, i.e., if $x \equiv x'$ and $x \leq y$, then $x' \leq y$, similarly, if $x \equiv x'$ and $y \leq x$, then $y \leq x'$.

Let $X/{\equiv}$ be the set of $\equiv$-equivalence classes; if $[x], [y] \in X/{\equiv}$, define $[x] \leq [y]$ if and only if $x \leq y$ (why is this well defined?). Prove that $(X/{\equiv}, \leq)$ is a partially ordered set.

(51) Show that $\varnothing$ and $\mathbb{W}$ are both minimal in the order $\leq_{\mathrm{m}}$, incomparable in $\leq_{\mathrm{m}}$, and that $\{\varnothing\}$ and $\{\mathbb{W}\}$ are $\equiv_{\mathrm{m}}$-equivalence classes.

(52) Assume that $X \neq \mathbb{B} \neq Y$ and $X \neq \varnothing \neq Y$. Show that the *Turing join* $X \oplus Y$ is the least upper bound of X and Y with respect to $\leq_{\mathrm{m}}$ (i.e., if $X, Y \leq_{\mathrm{m}} Z$, then $X \oplus Y \leq_{\mathrm{m}} Z$).

(53) Show that a set $X \subseteq \mathbb{W}^k$ is Π_1 if and only if there is a computable set $Y \subseteq \mathbb{W}^{k+1}$ such that for all $\vec{w} \in \mathbb{W}^k$, we have

$$\vec{w} \in X \iff \forall v (\vec{w}, v) \in Y.$$

Use this to show that $\mathbf{Emp} \equiv_{\mathrm{m}} \mathbb{W} \backslash \mathbf{K}$.

[*Hint.* The set $\mathbb{W} \backslash \mathbf{K}$ is Π_1-complete. Why?]

(54) Prove that $\mathbf{K}$ is not an index set.

(55) Prove that $\mathbf{Inf}$ and $\mathbf{Tot}$ are neither Σ_1 nor Π_1.

(56) Let $g : \mathbb{W}^k \to \mathbb{W}$ be a total computable function. Consider $\mathbf{Eq}(g) := \{w\,;\, f_{w,k} = g\}$ and show that $\mathbf{Tot} \leq_{\mathrm{m}} \mathbf{Eq}(g)$.