Automata & Formal Languages
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Part II of the Mathematical Tripos
University of Cambridge
Prof. Dr. B. Löwe

Example Sheet #3

(29) Consider the register machine M given by the following program:

$$q_{\mathrm{S}} \mapsto\ ?_{\mathbf{0}}(0, q_0, q_1) \qquad q_0 \mapsto\ +_{\mathbf{0}}(1, q_2) \qquad q_1 \mapsto\ +_{\mathbf{1}}(2, q_2)$$
$$q_2 \mapsto\ -(0, q_{\mathrm{H}}, q_{\mathrm{S}}) \qquad q_{\mathrm{H}} \mapsto\ ?_{\varepsilon}(0, q_{\mathrm{H}}, q_{\mathrm{H}})$$

Assume that the register content of the registers indexed 0, 1, and 2 is **001100**, **1100**, and **000**, respectively. Determine the first twelve snapshots of the computational sequence produced by this machine on the given input. Will the computation eventually halt? If so, what are the halting time and the register content at time of halting?

(30) Consider the register machine given by the following program. Determine the operation performed by this register machine.

$$q_{\mathrm{S}} \mapsto\ -(0, q_0, q_3) \qquad q_0 \mapsto\ +_{\mathbf{0}}(2, q_1) \qquad q_1 \mapsto\ +_{\mathbf{0}}(3, q_4) \qquad q_2 \mapsto\ +_{\mathbf{0}}(2, q_7)$$
$$q_3 \mapsto\ ?_{\varepsilon}(1, q_6, q_{\mathrm{H}}) \qquad q_4 \mapsto\ +_{\mathbf{0}}(4, q_0) \qquad q_5 \mapsto\ +_{\mathbf{0}}(4, q_3) \qquad q_6 \mapsto\ -(2, q_5, q_7)$$
$$q_7 \mapsto\ -(3, q_2, q_8) \qquad q_8 \mapsto\ ?_{\mathbf{0}}(4, q_9, q_3) \qquad q_9 \mapsto\ +_{\mathbf{0}}(2, q_7) \qquad q_{\mathrm{H}} \mapsto ?_{\varepsilon}(0, q_8, q_9)$$

[*Remark.* It helps to visualise the register machine with a diagram in the style of the graphical representations of automata.]

(31) Consider the following operation: given a word $w = a_0...a_n$, duplicate the final two letters, i.e., produce $a_0...a_{n-2}a_{n-1}a_{n-1}a_na_n$; if $w = \varepsilon$, don't do anything; if $w = a$, produce aa. Explicitly write down a register machine M_0 (giving all program lines in detail) that performs this operation.

Find machines M_1, M_2, and M_3 performing the same operation such that

(i) M_0 and M_1 are different, strongly equivalent, and have the same upper register index;

(ii) M_0 and M_2 are different, strongly equivalent, and have different upper register indices;

(iii) M_0 and M_3 are not strongly equivalent.

(32) Show that the question "*Is the content of registers i and j the same?*" can be answered by a register machine and that the operation "*Replace all occurrences of the* **0** *in register i by* **1**" can be performed by a register machine.

[In this example, you may use all operations performed and questions answered by register machines from § 4.2; in particular, you do not have to present a complete description of the program of the register machine.]

(33) We consider the following notions of equivalence between register machines M and M':

(i) The machines M and M' are *operation equivalent* if they perform the same operations.

(ii) The machines M and M' are *function equivalent* if for all k, they define the same partial functions on $\mathbb{W}^k$, i.e., if $f_{M,k} = f_{M',k}$.

(iii) The machines M and M' are *weakly equivalent* if they define the same languages, i.e., $\mathrm{W}_M = \mathrm{W}_{M'}$ (where $\mathrm{W}_M := \mathrm{dom}(f_{M,1})$).

Show that strong equivalence implies operation equivalence, operation equivalence implies function equivalence, function equivalence implies weak equivalence, and that none of the implications can be reversed.

(34) Let $X \subseteq \mathbb{B}^k$. Prove that X is finite if and only if every partial function f such that $\mathrm{dom}(f) = X$ is computable.

(35) Define the standard *lexicographic order* on $\mathbb{B}$: $w \leq_{\mathrm{lex}} w'$ if either $w \subseteq w'$ or, if k is the least number such that $w(k) \neq w'(k)$, then $w(k) = \mathbf{0} \neq \mathbf{1} = w'(k)$. Show that $(\mathbb{B}, \leq_{\mathrm{lex}})$ is not isomorphic to $(\mathbb{N}, \leq)$.

(36) Revisit the notions of splitting and merging of words and consider the words

$$w_0 := \mathbf{01010}, \qquad w_1 := \mathbf{110101}, \qquad w_2 := \mathbf{00111},$$
$$w_3 := \mathbf{100}, \qquad w_4 := \mathbf{111110}, \text{ and} \qquad w_5 := \mathbf{11111}.$$

Compute the following words:

(i) $w_0 * w_1$;
(ii) $(w_2)_{(0)}$;
(iii) $(w_3)_{(1)}$;
(iv) $(w_4)_{(0)} * (w_4)_{(1)}$;
(v) $s(w_4)$;
(vi) $s(w_5)$;
(vii) u such that $\#(u) = \#(w_4) + \#(w_5)$;
(viii) u such that $\#(u) = \#(w_3) \cdot \#(w_5)$.

(37) Suppose that $f : \mathbb{N} \to \mathbb{N}$ is a total and increasing function. Prove that if f is computable, then the range of f is computable.

(38) Let $f \colon \mathbb{N}^2 \to \mathbb{N}$ be any primitive recursive function. Show that the following numerical functions are primitive recursive:

$$g_0(n,m) := \begin{cases} 1 & n = m, \\ 0 & n \neq m; \end{cases} \qquad g_2(n,m) := \begin{cases} 1 & m|n, \\ 0 & \text{otherwise; and} \end{cases}$$
$$g_3(n,k) := \sum_{i=1}^{k} f(n,i); \qquad g_3(n) := \begin{cases} 1 & \text{if } n \text{ is prime,} \\ 0 & \text{otherwise.} \end{cases}$$

[*Hint.* Use the functions from Example 4.21 in the typed notes.]

(39) Let $b \colon \mathbb{B}^k \to \mathbb{B}$ be a total computable function and $R \subseteq \mathbb{B}^{k+1}$ be a computable set. We call $h \colon \mathbb{B}^k \to \{\mathbf{0}, \mathbf{1}\}$ the *bounded search function with bound b and query R* if

$$h(\vec{v}) = \begin{cases} \mathbf{1} & \text{there is } w \in \mathbb{B} \text{ such that } w \leq_{\mathrm{shortlex}} b(\vec{v}) \text{ and } (\vec{v}, w) \in R \text{ and} \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

Show that h is computable. Can you do this without having a bound function?

(40) Suppose $f \colon \mathbb{B}^{k+1} \dashrightarrow \mathbb{B}$ is a partial function, then the partial function h defined by

$$h(\vec{w}) := \begin{cases} v & \text{if for all } u \leq_{\mathrm{shortlex}} v, \text{ we have that } f(\vec{w}, u)\downarrow \text{ and} \\ & \quad v \text{ is } <_{\mathrm{shortlex}}\text{-minimal such that } f(\vec{w}, v) = \varepsilon \text{ or} \\ \uparrow & \text{otherwise} \end{cases}$$

is called the *minimisation result of f*. Prove that if f is computable, then so is h.

(41) Show that all noncontracting $L \subseteq \mathbb{B}$ are computable.