PART II AUTOMATA AND FORMAL LANGUAGES
MICHAELMAS 2021-22
EXAMPLE SHEET 4

Unless explicitly asked to, you need not *prove* that any machine, grammar or expression you construct defines the language you say it does. $*$ denotes a harder problem.

(1) Let G be the CFG given by

$$S \to ABS \mid AB, \quad A \to aA \mid a, \quad B \to bA$$

For each of the words $aabaab, aaaaba, aabbaa, abaaba$, determine whether or not they lie in $\mathcal{L}(G)$. If so, give a derivation and a parse tree; if not, explain why not.

(2) Convert the following CFG to CNF, giving a justification for your answer.

$$S \to aSbb \mid T, \quad T \to bTaa \mid S \mid \epsilon$$

(3) Give a CFG for each of the following CFL's, and then transform each such CFG into CNF (giving a justification for the transformation).

(a) $\{a^n b^{2n} c^k \mid k, n \geq 1\}$

(b) $\{a^n b^k a^n \mid k, n \geq 1\}$

(c) $\{a^k b^m c^n \mid k, m, n \geq 1,\ 2k \geq n\}$

(4) For each of the following languages, either show that it is a CFL by constructing a CFG for it, or use the pumping lemma to show that it is not a CFL:

(a) $\{a^n b^m \mid n \neq m\}$

(b) $\{a^m b^n c^m d^n \mid m, n \geq 1\}$

(c) $\{a^n b^m c^k d^l \mid 2n = 3m \text{ and } 5k = 7l\}$

(d) $\{a^n b^m c^k d^l \mid 2n = 3k \text{ and } 5m = 7l\}$

(e) $\{a^n b^m c^k d^l \mid 2n = 3k \text{ or } 5m = 7l\}$

(f) $\{1^p \mid p \text{ is prime}\}$

(g) $\{a^{2^n} \mid n \geq 1\}$

(h) $\{ww \mid w \in \{a, b\}^*\}$

(i*) $\{a, b\}^* \setminus \{ww \mid w \in \{a, b\}^*\}$

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(5) Let G be the CFG grammar generated by the CFG $S \to aSb \mid \epsilon$. In the lectures it was asserted that $\mathcal{L}(G) = \{a^n b^n \mid n \geq 0\}$. Prove this statement.

(6) Let $G = (N, \Sigma, P, S)$ be a CFG in CNF. Suppose we form a new CFG G' from G by adding, for each production of the form $B \to a$ in P (where $a \in \Sigma$), the production $B \to \epsilon$. Describe the new language $\mathcal{L}(G')$ in terms of the original language $\mathcal{L}(G)$, giving an argument for your answer.

(7) Let L, M be CFL's, and let a be any symbol. Show that the following are all CFL's: $\emptyset$, $\{\epsilon\}$, $\{a\}$, $L \cup M$, LM, and L^*.
Conclude that every regular language is a CFL.

(8) Give a CFG which generates the set of regular expressions over the alphabet $\{0, 1\}$. Take as the set of terminals $\Sigma = \{0, 1, (,), +, ^*, \emptyset, \epsilon\}$.
Show that this language is not regular. (*Hint: look at the brackets!*)

(9) (a) Show that the following two languages are both CFL's:
$L_1 := \{a^n b^n c^i \mid n, i \geq 1\}$, and $L_2 := \{a^i b^n c^n \mid n, i \geq 1\}$.

(b) Show that the language $L := \{a^n b^n c^n \mid n \geq 1\}$ is not a CFL.

(c) Show that $L_1 \cap L_2 = L$, and hence the intersection of two CFL's isn't always a CFL.

(d) Conclude that the complement of a CFL need not be a CFL.

(10) (a) Let G be a CFG in CNF, and $w \in \mathcal{L}(G)$ a word of length $n \geq 1$. Show that *any* derivation of w in G uses precisely $2n - 1$ steps.

(b) Let G be a CFG in CNF with m nonterminals. Show that if $\mathcal{L}(G) \neq \emptyset$, then $\mathcal{L}(G)$ contains at least one word of length $< 2^{m+1}$.

(c) Let G be a CFG in CNF with m nonterminals. Show that if $\mathcal{L}(G)$ contains a word of length $\geq 2^{m+1}$, then $\mathcal{L}(G)$ is infinite.

(d*) Give an algorithm that, on input of a CFG G and a word w on the terminal symbols of G, decides if $w \in \mathcal{L}(G)$ or not.

(e*) Give an algorithm that, on input of a CFG G, decides if $\mathcal{L}(G) = \emptyset$ or not.

(11*) Suppose that D is a DFA and M is a NPDA. Define a notion of product automaton for D and M and show that this is an NPDA. Use this construction to prove that the intersection of a regular language with a CFL is a CFL.