

**PART II AUTOMATA AND FORMAL LANGUAGES**  
**MICHAELMAS 2019-20**  
**EXAMPLE SHEET 4**

Unless explicitly asked to, you need not *prove* that any machine, grammar or expression you construct defines the language you say it does. \* denotes a harder problem.

- (1) Let  $G$  be the CFG given by

$$S \rightarrow ABS \mid AB, \quad A \rightarrow aA \mid a, \quad B \rightarrow bA$$

For each of the words  $aabaab, aaaaba, aabbaa, abaaba$ , determine whether or not they lie in  $\mathcal{L}(G)$ . If so, give a derivation and a parse tree; if not, explain why not.

- (2) Convert the following CFG to CNF, giving a justification for your answer.

$$S \rightarrow aSbb \mid T, \quad T \rightarrow bTaa \mid S \mid \epsilon$$

- (3) Give a CFG for each of the following CFL's, and then transform each such CFG into CNF (giving a justification for the transformation).

(a)  $\{a^n b^{2n} c^k \mid k, n \geq 1\}$

(b)  $\{a^n b^k a^n \mid k, n \geq 1\}$

(c)  $\{a^k b^m c^n \mid k, m, n \geq 1, 2k \geq n\}$

- (4) For each of the following languages, either show that it is a CFL by constructing a CFG for it, or use the pumping lemma to show that it is not a CFL:

(a)  $\{a^n b^m \mid n \neq m\}$

(b)  $\{a^m b^n c^m d^n \mid m, n \geq 1\}$

(c)  $\{a^n b^m c^k d^l \mid 2n = 3m \text{ and } 5k = 7l\}$

(d)  $\{a^n b^m c^k d^l \mid 2n = 3k \text{ and } 5m = 7l\}$

(e)  $\{a^n b^m c^k d^l \mid 2n = 3k \text{ or } 5m = 7l\}$

(f)  $\{1^p \mid p \text{ is prime}\}$

(g)  $\{a^{2^n} \mid n \geq 1\}$

(h)  $\{ww \mid w \in \{a, b\}^*\}$

(i\*)  $\{a, b\}^* \setminus \{ww \mid w \in \{a, b\}^*\}$

- (5) Let  $G$  be the CFG grammar generated by the CFG  $S \rightarrow aSb \mid \epsilon$ . In the lectures it was asserted that  $\mathcal{L}(G) = \{a^n b^n \mid n \geq 0\}$ . Prove this statement.
- (6) Let  $G = (N, \Sigma, P, S)$  be a CFG in CNF. Suppose we form a new CFG  $G'$  from  $G$  by adding, for each production of the form  $B \rightarrow a$  in  $P$  (where  $a \in \Sigma$ ), the production  $B \rightarrow \epsilon$ . Describe the new language  $\mathcal{L}(G')$  in terms of the original language  $\mathcal{L}(G)$ , giving an argument for your answer.
- (7) Let  $L, M$  be CFL's, and let  $a$  be any symbol. Show that the following are all CFL's:  $\emptyset$ ,  $\{\epsilon\}$ ,  $\{a\}$ ,  $L \cup M$ ,  $LM$ , and  $L^*$ .  
Conclude that every regular language is a CFL.
- (8) Give a CFG which generates the set of regular expressions over the alphabet  $\{0, 1\}$ . Take as the set of terminals  $\Sigma = \{0, 1, (, ), +, *, \emptyset, \epsilon\}$ .  
Show that this language is not regular. (*Hint: look at the brackets!*)
- (9) (a) Show that the following two languages are both CFL's:  
 $L_1 := \{a^n b^n c^i \mid n, i \geq 1\}$ , and  $L_2 := \{a^i b^n c^n \mid n, i \geq 1\}$ .  
(b) Show that the language  $L := \{a^n b^n c^n \mid n \geq 1\}$  is not a CFL.  
(c) Show that  $L_1 \cap L_2 = L$ , and hence the intersection of two CFL's isn't always a CFL.  
(d) Conclude that the complement of a CFL need not be a CFL.
- (10) (a) Let  $G$  be a CFG in CNF, and  $w \in \mathcal{L}(G)$  a word of length  $n \geq 1$ . Show that *any* derivation of  $w$  in  $G$  uses precisely  $2n - 1$  steps.  
(b) Let  $G$  be a CFG in CNF with  $m$  nonterminals. Show that if  $\mathcal{L}(G) \neq \emptyset$ , then  $\mathcal{L}(G)$  contains at least one word of length  $< 2^{m+1}$ .  
(c) Let  $G$  be a CFG in CNF with  $m$  nonterminals. Show that if  $\mathcal{L}(G)$  contains a word of length  $\geq 2^{m+1}$ , then  $\mathcal{L}(G)$  is infinite.  
(d\*) Give an algorithm that, on input of a CFG  $G$  and a word  $w$  on the terminal symbols of  $G$ , decides if  $w \in \mathcal{L}(G)$  or not.  
(e\*) Give an algorithm that, on input of a CFG  $G$ , decides if  $\mathcal{L}(G) = \emptyset$  or not.
- (11\*) Suppose that  $D$  is a DFA and  $M$  is a NPDA. Define a notion of product automaton for  $D$  and  $M$  and show that this is an NPDA. Use this construction to prove that the intersection of a regular language with a CFL is a CFL.