Applied probability, Lent 2025. ss2871@cam.ac.uk

Example Sheet 3

1. Compute the average busy period for a $M/M/\infty$ and a $M/M/1$ queue. (The busy period B is the length of time between the arrival of the first customer and the first time afterwards that all servers are free).

2. Consider the $M/M/n$ queue, where the arrival rate is λ and the service rate in each queue is μ. For which values of the parameters is the queue length transient, positive recurrent and null recurrent? Compute the invariant distribution when there exists one.

3. *Queue with impatient customers.* Customers arrive at a single server at rate λ and require an exponential amount of service with rate μ. Customers waiting in line are impatient and if they are not in service they will leave at rate δ, independent of their position in the queue. (a) Show that for any $\delta > 0$ the system has an invariant distribution. (b) Find the invariant distribution when $\delta = \mu$.

4. Is the time-reversal of a tandem of $M/M/1$ queues reversible at equilibrium? If not can you describe the time-reversal? Find the distribution of the process D_t which counts the number of customers whose service is completed by time t.

5. Consider the following queue. Customers arrive at rate $\lambda > 0$ and are served by one server at rate μ. After service, each customer returns to the beginning of the queue with probability $p \in (0, 1)$. Let $(L_t)_{t\geq 0}$ denote the queue length. Show that L has the same distribution as a M/M/1 queue with modified rates. For which parameters is L transient, and for which is it recurrent?

6. Let $(L_t)_{t\geq 0}$ denotes the length of an $M/M/1$ queue with rates $\lambda < \mu$. Let π denote the equilibrium distribution. Let $(D_t)_{t\geq 0}$ denotes the departure process from the queue. By considering all possibilities leading to the events below, show directly that as $h \to 0$,

$$\mathbb{P}_\pi(D_h - D_0 = 0) = 1 - \lambda h + o(h)$$

and that

$$\mathbb{P}_\pi(D_h - D_0 \geq 1) = \lambda h + o(h).$$

What have you proved?

7. Prove that the traffic equations for a Jackson network have a unique solution.

8. Let $X_t = (X_t^1, \dots, X_t^N, t \geq 0)$ denote a Jackson network of N queues, with arrival rate λ_i and service rate μ_i in queue i, and each customer moves to queue $j \neq i$ with probability p_{ij} after service from queue i. We assume $\sum_j p_{ij} < 1$ for each $i = 1, \dots, N$ and that the traffic equations have a solution such that $\bar{\lambda}_i < \mu_i$.

Describe the time-reversal of X at equilibrium.

Let $D_i(t)$ be the process of (final) departures from queue i. Show that, at equilibrium, $(D_i(t), t \geq 0)_{1\leq i\leq N}$ are independent Poisson processes and specify the rates. Show further that X_t is independent $(D_i(s), 1 \leq i \leq N, 0 \leq s \leq t)$.

9. Consider a system of N queues serving a finite number K of customers. The system evolves as follows. At station $1 \leq i \leq N$, customers are served one at a time at rate μ_i. After service, each customer moves to queue j with probability $p_{ij} > 0$. We assume here that the system is closed, ie, $\sum_j p_{ij} = 1$ for all $1 \le i \le N$.

Let $S = \{(n_1, \ldots, n_N) : n_i \in \mathbb{N}, \sum_{i=1}^N n_i = K\}$ be the state space of the Markov chain. Write down its Q-matrix. Also write down the Q-matrix R corresponding to the position in the network of one customer (that is, when $K = 1$). Show that there is a unique distribution $(\lambda_i)_{1 \le i \le n}$ such that $\lambda R = 0$. Show that

$$\pi(n) = C_N \prod_{i=1}^N \lambda_i^{n_i}, n \in S$$

defines an invariant measure for the chain. Are the queue lengths independent at equilibrium?

10. Kafkaian Insurances Inc. has a peculiar way of processing claims. Claims arrive at a rate of 10 per day, and are initially randomly assigned to one of two departments, respectively D_1 and D_2. The service rates in D_1 and D_2 are $\mu_1 = 15$ and $\mu_2 = 20$ per day, respectively. After looking at each claim, the relevant department settles the claim with probability 1/2, and otherwise finds a pretext to hand it over to the other department to process it. This goes on until the claim is finally settled by one of the two departments.

(a) What proportion of claims is finally settled by D_1?

(b) How many claims are settled on average every month by Kafkaian Insurances Inc.?

(c) The manager of the company wants to reward the work of his employees based on the number of claims that their department settles. Is that a good idea?

11. Consider a G/M/1 queueing system: the nth client arrives at time $A_n = \sum_{i=1}^n \xi_i$, where (ξ_i) is a sequence of nonnegative i.i.d. random variables, and the service times are i.i.d. exponential with rate μ. Let $X_n = L(A_n)$ be the size of the queue just before the nth arrival.

(i) Show that (X_n) is a discrete-time Markov chain, and specify its transition matrix.

(ii) Show that if $\rho := (\mu \mathbb{E} A)^{-1} < 1$ then the chain (X_n) has a unique equilibrium distribution $\pi = (\pi_i)$ and hence is positive recurrent. Here

$$\pi_i = (1 - \eta)\eta^i, \quad i = 0, 1, \ldots$$

and $\eta \in (0, 1)$ is a solution to $\eta = \phi(\mu(\eta - 1))$, where for $\theta \in \mathbb{R}$, $\phi(\theta) = \mathbb{E}(e^{\theta \xi})$.

12. Consider the square lattice $\mathbb{Z}^2$, and endow each site $x \in \mathbb{Z}^2$ with a weight W_x, which is an independent exponential random variable of rate μ. An oriented path π between $(1, 1)$ and a point (M, N), with $M, N \ge 1$, is called increasing if it only ever goes in the North and East directions. Define the weight of an increasing path π to be $W(\pi) = \sum_{x \in \pi} W_x$, and the passage time from $(1, 1)$ to (M, N) to be

$$T(M, N) = \max_\pi W(\pi)$$

where the max is over increasing π's from $(1, 1)$ to (M, N). This model is called *Last Passage Percolation*. [Simulations showing optimal paths from (0,0) are interesting.]

The goal of this question is to relate this model to a sequence of N queues operating under the following protocol. At time 0 there are M customers in the first queue, and none at any other queue. Customers are served one at a time at rate μ in each queue, and after service at queue i, a customer moves on to queue $i + 1$. Customers leave the system for good after being served at queue N. Let $\tau(M, N)$ denote the time at which the Mth customer completes service in queue N. Show that $\tau(M, N)$ and $T(M, N)$ have the same distribution.