

Example Sheet 3

1. Compute the average busy period for a $M/M/\infty$ and a $M/M/1$ queue. (The busy period B is the length of time between the arrival of the first customer and the first time afterwards that all servers are free).
2. Consider the $M/M/n$ queue, where the arrival rate is λ and the service rate in each queue is μ . For which values of the parameters is the queue length transient, positive recurrent and null recurrent? Compute the invariant distribution when there exists one.
3. *Queue with impatient customers.* Customers arrive at a single server at rate λ and require an exponential amount of service with rate μ . Customers waiting in line are impatient and if they are not in service they will leave at rate δ , independent of their position in the queue. (a) Show that for any $\delta > 0$ the system has an invariant distribution. (b) Find the invariant distribution when $\delta = \mu$.
4. Is the time-reversal of a tandem of $M/M/1$ queues reversible at equilibrium? If not can you describe the time-reversal? Find the distribution of the process D_t which counts the number of customers whose service is completed by time t .
5. Consider the following queue. Customers arrive at rate $\lambda > 0$ and are served by one server at rate μ . After service, each customer returns to the beginning of the queue with probability $p \in (0, 1)$. Let $(L_t)_{t \geq 0}$ denote the queue length. Show that L has the same distribution as a $M/M/1$ queue with modified rates. For which parameters is L transient, and for which is it recurrent?
6. Let $(L_t)_{t \geq 0}$ denotes the length of an $M/M/1$ queue with rates $\lambda < \mu$. Let π denote the equilibrium distribution. Let $(D_t)_{t \geq 0}$ denotes the departure process from the queue. By considering all possibilities leading to the events below, show directly that as $h \rightarrow 0$,

$$\mathbb{P}_\pi(D_h - D_0 = 0) = 1 - \lambda h + o(h)$$

and that

$$\mathbb{P}_\pi(D_h - D_0 \geq 1) = \lambda h + o(h).$$

What have you proved?

7. Prove that the traffic equations for a Jackson network have a unique solution.
8. Let $X_t = (X_t^1, \dots, X_t^N, t \geq 0)$ denote a Jackson network of N queues, with arrival rate λ_i and service rate μ_i in queue i , and each customer moves to queue $j \neq i$ with probability p_{ij} after service from queue i . We assume $\sum_j p_{ij} < 1$ for each $i = 1, \dots, N$ and that the traffic equations have a solution such that $\bar{\lambda}_i < \mu_i$.

Describe the time-reversal of X at equilibrium.

Let $D_i(t)$ be the process of (final) departures from queue i . Show that, at equilibrium, $(D_i(t), t \geq 0)_{1 \leq i \leq N}$ are independent Poisson processes and specify the rates. Show further that X_t is independent $(D_i(s), 1 \leq i \leq N, 0 \leq s \leq t)$.

9. Consider a system of N queues serving a finite number K of customers. The system evolves as follows. At station $1 \leq i \leq N$, customers are served one at a time at rate μ_i . After

service, each customer moves to queue j with probability $p_{ij} > 0$. We assume here that the system is closed, ie, $\sum_j p_{ij} = 1$ for all $1 \leq i \leq N$.

Let $S = \{(n_1, \dots, n_N) : n_i \in \mathbb{N}, \sum_{i=1}^N n_i = K\}$ be the state space of the Markov chain. Write down its Q -matrix. Also write down the Q -matrix R corresponding to the position in the network of one customer (that is, when $K = 1$). Show that there is a unique distribution $(\lambda_i)_{1 \leq i \leq N}$ such that $\lambda R = 0$. Show that

$$\pi(n) = C_N \prod_{i=1}^N \lambda_i^{n_i}, n \in S$$

defines an invariant measure for the chain. Are the queue lengths independent at equilibrium?

10. Kafkaian Insurances Inc. has a peculiar way of processing claims. Claims arrive at a rate of 10 per day, and are initially randomly assigned to one of two departments, respectively D_1 and D_2 . The service rates in D_1 and D_2 are $\mu_1 = 15$ and $\mu_2 = 20$ per day, respectively. After looking at each claim, the relevant department settles the claim with probability $1/2$, and otherwise finds a pretext to hand it over to the other department to process it. This goes on until the claim is finally settled by one of the two departments.

- (a) What proportion of claims is finally settled by D_1 ?
- (b) How many claims are settled on average every month by Kafkaian Insurances Inc.?
- (c) The manager of the company wants to reward the work of his employees based on the number of claims that their department settles. Is that a good idea?

11. Consider a G/M/1 queueing system: the n th client arrives at time $A_n = \sum_{i=1}^n \xi_i$, where (ξ_i) is a sequence of nonnegative i.i.d. random variables, and the service times are i.i.d. exponential with rate μ . Let $X_n = L(A_n)$ be the size of the queue just before the n th arrival.

- (i) Show that (X_n) is a discrete-time Markov chain, and specify its transition matrix.
- (ii) Show that if $\rho := (\mu \mathbb{E}A)^{-1} < 1$ then the chain (X_n) has a unique equilibrium distribution $\pi = (\pi_i)$ and hence is positive recurrent. Here

$$\pi_i = (1 - \eta)\eta^i, \quad i = 0, 1, \dots$$

and $\eta \in (0, 1)$ is a solution to $\eta = \phi(\mu(\eta - 1))$, where for $\theta \in \mathbb{R}$, $\phi(\theta) = \mathbb{E}(e^{\theta\xi})$.

12. Consider the square lattice \mathbb{Z}^2 , and endow each site $x \in \mathbb{Z}^2$ with a weight W_x , which is an independent exponential random variable of rate μ . An oriented path π between $(1, 1)$ and a point (M, N) , with $M, N \geq 1$, is called increasing if it only ever goes in the North and East directions. Define the weight of an increasing path π to be $W(\pi) = \sum_{x \in \pi} W_x$, and the passage time from $(1, 1)$ to (M, N) to be

$$T(M, N) = \max_{\pi} W(\pi)$$

where the max is over increasing π 's from $(1, 1)$ to (M, N) . This model is called *Last Passage Percolation*. [Simulations showing optimal paths from $(0, 0)$ are interesting.]

The goal of this question is to relate this model to a sequence of N queues operating under the following protocol. At time 0 there are M customers in the first queue, and none at any other queue. Customers are served one at a time at rate μ in each queue, and after service at queue i , a customer moves on to queue $i + 1$. Customers leave the system for good after being served at queue N . Let $\tau(M, N)$ denote the time at which the M th customer completes service in queue N . Show that $\tau(M, N)$ and $T(M, N)$ have the same distribution.