Lent 2026 **Analysis of Functions – Example Sheet 1** P.P. Varjú

1. Let μ be a complex measure on a measurable space $(E, \mathcal{A})$. Define the set function

$$|\mu|(A) = \sup\Big\{\sum_{i=1}^{\infty} |\mu(A_i)| : A_i \subset A \text{ pairwise disjoint}\Big\}$$

for $A \in \mathcal{A}$.

(a) Show that $|\mu|$ is a finite positive measure. (It is called the total variation measure.)

(b) Show that $\|\mu\| := |\mu|(E)$ is a norm on the vector space of complex measures on $(E, \mathcal{A})$. (This is called the total variation norm.)

(c*) Show that the vector space of complex measures endowed with this norm is a Banach space.

Hint: To show that $|\mu|(A) < \infty$, *use the Hahn decomposition separately for* $\mathrm{Re}(\mu)$ *and* $\mathrm{Im}(\mu)$.

2. Let μ be a complex measure on a measurable space $(E, \mathcal{A})$. Show that there is a measurable function $f : E \to \mathbf{C}$ such that $|f(x)| = 1$ for $|\mu|$-almost all x, and

$$\mu(A) = \int_A f(x) d|\mu|(x)$$

for all $A \in \mathcal{A}$.

3. Let $X_1, X_2, \ldots$ be a sequence of independent random variables satisfying $\mathbf{P}(X_i = 0) = 1/3$ and $\mathbf{P}(X_i = 1) = 2/3$. Let ν be the distribution of the random variable $\sum_{i=1}^{\infty} X_i 2^{-i}$. That is, ν is the probability measure on $\mathbf{R}$ with the property

$$\nu(A) = \mathbf{P}\Big(\sum_{i=1}^{\infty} X_i 2^{-i} \in A\Big)$$

for all measurable sets A. Show that ν is singular with respect to the Lebesgue measure.

Hint: Consider a variant μ *of* ν *that we obtain when the probabilities* $1/3, 2/3$ *are replaced by* $1/2, 1/2$. *Show that* ν *is the restriction of the Lebesgue measure to* $[0, 1]$.

4. Suppose $f : \mathbf{R} \to \mathbf{C}$ is integrable and let $F(x) = \int_{-\infty}^{x} f(t)dt$. Show that F is differentiable with $F'(x) = f(x)$ at each Lebesgue point $x \in \mathbf{R}$. Deduce that F is differentiable almost everywhere.

Let $F : \mathbf{R} \to [0, 1]$ be a monotone non-decreasing continuous function that is differentiable almost everywhere and $\lim_{x\to-\infty} F(x) = 0$. Is it true that

$$F(x) = \int_{-\infty}^{x} F'(t)dt$$

for all $x \in \mathbf{R}$?

5. Suppose $\varphi \in L^\infty(\mathbf{R}^n)$ satisfies $\operatorname{supp} \varphi \subset B_1(0)$ and $\int_{\mathbf{R}^n} \varphi \, dx = 1$. Set $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(\varepsilon^{-1}x)$. Show that if $f \in L^1(\mathbf{R}^n)$, and x is a Lebesgue point of f, then

$$\varphi_\varepsilon * f(x) \to f(x), \qquad \text{as } \varepsilon \to 0.$$

6. Let ν be a finite measure on $\mathbf{R}$. Prove that the limit

$$f(x) = \lim_{r \searrow 0} \frac{\nu([x-r, x+r])}{2r}$$

exists for Lebesgue almost all x. Prove that f is the Radon-Nikodym derivative of ν_{ac} in the Lebesgue decomposition of ν with respect to the Lebesgue measure.

[*Hint:* For a finite measure ν on $\mathbf{R}$, define the maximal function

$$M\nu(x) = \sup_{r>0} \frac{\nu([x-r, x+r])}{2r}.$$

The proof that was given for the Hardy-Littlewood maximal function in the lectures, yields the maximal inequality $|\{x : M\nu(x) > t\}| \leq 5t^{-1}\|\nu\|$, which you may use without proof. You may find this useful to prove that $f(x) \equiv 0$ if ν is singular with respect to the Lebesgue measure.]

7. Show that for all $\varepsilon > 0$, there is a measurable set $A_\varepsilon \subset [0,1]$ such that $0.1|I| \leq |A_\varepsilon \cap I| \leq 0.9|I|$ for all intervals I of length at least ε.

Show that there is no measurable set A that satisfies this property for all $\varepsilon > 0$.

8. Let $C_0(\mathbf{R})$ be the space of continuous $\mathbf{R} \to \mathbf{C}$ functions that vanish at infinity, that is, $\lim_{|x|\to\infty} f(x) = 0$ for all $f \in C_0(\mathbf{R})$. This is a Banach space with respect to the norm $\|f\| := \sup_{x\in\mathbf{R}} |f(x)|$. Using the Riesz representation theorem about the dual of the space of continuous functions on a compact metric space, prove that for every bounded linear functional $L : C_0(\mathbf{R}) \to \mathbf{R}$, there is a complex Borel measure μ on $\mathbf{R}$ such that

$$L(f) = \int f d\mu$$

for all $f \in C_0(\mathbf{R})$.

9. (a) Suppose that $f \in L^{p_0}(E,\mu) \cap L^{p_1}(E,\mu)$ with $1 \leq p_0 < p_1 \leq \infty$. For $0 \leq \theta \leq 1$, define p_θ by

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

Show that $f \in L^{p_\theta}(E,\mu)$ with

$$\|f\|_{p_\theta} \leq \|f\|_{p_0}^{1-\theta} \|f\|_{p_1}^{\theta}.$$

(b) Show that for $p_1 \neq p_2$ we have $L^{p_1}(\mathbf{R}^n) \not\subset L^{p_2}(\mathbf{R}^n)$. For which p_1, p_2 do we have $L^{p_1}_{loc}(\mathbf{R}^n) \subset L^{p_2}_{loc}(\mathbf{R}^n)$?

10. Let $F : \mathbf{R}^2 \to \mathbf{R}_{\geq 0}$ be a measurable function. Prove that

$$\Big(\int \Big(\int F(x,y)^{p_1} dx\Big)^{p_2/p_1} dy\Big)^{1/p_2} \leq \Big(\int \Big(\int F(x,y)^{p_2} dy\Big)^{p_1/p_2} dx\Big)^{1/p_1}$$

for all $0 < p_1 \leq p_2 < \infty$.

11. Let $I = (0,1)$ and $1 \leq p < \infty$. Exhibit a sequence $(f_j)_{j=1}^\infty$ with $f_j \in L^p(I)$ such that $f_j \to 0$ in $L^p(I)$, but $f_j(x)$ does not converge for any x. Does such a sequence exist if $p = \infty$?

12. Suppose $1 \leq p < \infty$.

(a) Suppose $f \in L^p(\mathbf{R}^n)$. Show that

$$|\{x : |f(x)| > \lambda\}| \leq \frac{\|f\|_p^p}{\lambda^p}.$$

This is known as Tchebychev's inequality, the $p = 1$ case is Markov's inequality.

(b) We say that a measurable $f : \mathbf{R}^n \to \mathbf{C}$ is in *weak-*$L^p(\mathbf{R}^n)$, written $f \in L^{p,w}(\mathbf{R}^n)$ if there exists a constant C such that

$$|\{x : |f(x)| > \lambda\}| \leq \frac{C^p}{\lambda^p}.$$

Show that $L^p(\mathbf{R}^n) \subset L^{p,w}(\mathbf{R}^n)$, and that the inclusion is proper.

13. Suppose that $f \in L^r(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$ for some $1 \leq r < \infty$. Show that $\|f\|_\infty = \lim_{p\to\infty} \|f\|_p$.

[Hint: you may find the estimates in Questions 9 and 12 useful.]