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Exercise 1. Suppose $f \in L^p_{loc}(\mathbb{R}^n)$ is a periodic function and let:

$$q = \left\{ x \in \mathbb{R}^n : |x_j| < \frac{1}{2}, j = 1, \dots, n \right\}.$$

Show that for any $\epsilon > 0$ there exists a smooth periodic function f_ϵ such that $\|f - f_\epsilon\|_{L^p(q)} < \epsilon$.

Exercise 2. Show that the series $S(x) = \sum_{k \in \mathbb{Z}^n} e^{-|x+k|^2/2}$ converges for every $x \in (0,1]^n$ and calculate the Fourier coefficients of S.

Exercise 3. Suppose that $\Omega \subset \mathbb{R}^n$ is open and bounded, let $f \in C^\infty_c(\Omega)$, and suppose $0 < \epsilon < 1$.

a) Show that $\int_\Omega (|f|^2 + \epsilon)^{\frac{p}{2}} dx \to \|f\|^p_{L^p}$ as $\epsilon \to 0$.

b) By considering $\int_\Omega (|f|^2 + \epsilon)^{\frac{p}{2}} dx = \int_{\mathbb{R}^n} \left(\frac{1}{n} \text{div}\, x\right) (|f|^2 + \epsilon)^{\frac{p}{2}} dx$, or otherwise, show that there exists a constant C, depending on Ω, p but not on f, such that $\|f\|_{L^p} \le C \|Df\|_{L^p}$.

Exercise 4. Let $s \in \mathbb{R}$.

a) Show that $\mathcal{S}$ is a dense subset of $H^s(\mathbb{R}^n)$.

b) Find a condition on s such that $\delta_x \in H^s(\mathbb{R}^n)$.

c) Show that $H^t(\mathbb{R}^n)$ is continuously embedded in $H^s(\mathbb{R}^n)$ for $s < t$.

d) Show that the derivative D^α is a bounded linear map from $H^{s+k}(\mathbb{R}^n)$ into $H^s(\mathbb{R}^n)$, where $k = |\alpha|$.

e) (*) Show that the pairing $\langle , \rangle : H^{-s}(\mathbb{R}^n) \times H^s(\mathbb{R}^n) \to \mathbb{C}$, which acts on $f \in H^{-s}(\mathbb{R}^n)$, $g \in H^s(\mathbb{R}^n)$ by

$$\langle f, g \rangle = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\xi) \hat{g}(\xi) d\xi$$

is well defined, and show that the map $g \mapsto \langle f, g \rangle$ is a bounded linear operator on $H^s(\mathbb{R}^n)$. Deduce that $H^s(\mathbb{R}^n)'$ may be identified with $H^{-s}(\mathbb{R}^n)$, and that $\mathcal{S}(\mathbb{R}^n) \subset H^s(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$ for all s.

Exercise 5. For two probability measures μ, ν on $\mathbb{R}^n$ their convolution $\mu * \nu$ is defined via

$$\mu * \nu(\phi) = \int_{\mathbb{R}^n} \phi(x+y) d\mu(x) d\nu(y), \phi \in \mathcal{S}(\mathbb{R}^n).$$

i) Show that $\mu * \nu$ defines an element $T_{\mu*\nu}$ of $\mathcal{S}'(\mathbb{R}^n)$ and find $\hat{T}_{\mu*\nu}$.

ii) Let $n = 1$ and let μ be a probability measure on $\mathbb{R}$ such that $\int_{\mathbb{R}} x d\mu(x) = 0$, $\int_{\mathbb{R}} x^2 d\mu(x) = 1$. Denote by $\mu^{*k} = \mu * \cdots * \mu$ the k-fold convolution (with $k \in \mathbb{N}$ factors) and define a Borel measure on $\mathbb{R}$

$$\lambda_k(A) = \mu^{*k}(\sqrt{k} A), \quad \text{where } \sqrt{k} A = \{\sqrt{k} x : x \in A\}, \ \ A \subseteq \mathbb{R} \text{ Borel}.$$

Show that the corresponding distributions T_{λ_k} converge in $\mathcal{S}'(\mathbb{R})$ as $k \to \infty$ and identify the limit.

iii) (*) Show that T_{λ_k} defines a sequence in $H^{-s}(\mathbb{R})$ whenever $s > 1/2$ and that it converges in this space. What if $s \le 1/2$?

Exercise 6. a) Suppose $s = \frac{n}{2} + \gamma$ for some $0 < \gamma < 1$. Show that there exists a constant $C_{n,\gamma} > 0$ such that for all $x, y \in \mathbb{R}^n$:

$$\int_{\mathbb{R}^n} \frac{|e^{ix\cdot\xi} - e^{iy\cdot\xi}|^2}{|\xi|^{2s}} d\xi \leqslant C_{n,\gamma} |x - y|^{2\gamma}$$

b) Show that if $s = \frac{n}{2} + k + \gamma$ for some $k \in \mathbb{Z}_{\geqslant 0}$, $0 < \gamma < 1$, then

$$H^s(\mathbb{R}^n) \subset C^{k,\gamma}(\mathbb{R}^n).$$

Exercise 7. Fix $s \in \mathbb{R}$, and suppose that $f \in H^s(\mathbb{R}^n)$.

a) Show that there exists a unique $u \in H^{s+4}(\mathbb{R}^n)$ which solves:

$$\Delta^2 u + u = f.$$

b) Show further that there exists $C > 0$ such that $\|u\|_{H^{s+4}} \leqslant C \|f\|_{H^s}$.

c) For what values of s does the equation hold in the sense of classical derivatives (possibly after redefining u, f on a set of measure zero)?

Exercise 8. Assume $s > \frac{1}{2}$ and suppose $u \in \mathcal{S}(\mathbb{R}^n)$. Define $Tu \in \mathcal{S}(\mathbb{R}^{n-1})$ by:

$$Tu(x') = u(x', 0), \qquad x' \in \mathbb{R}^{n-1}.$$

a) Show that if $\xi' \in \mathbb{R}^{n-1}$:

$$\widehat{Tu}(\xi') = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{u}(\xi', \xi_n) d\xi_n.$$

b) Deduce that:

$$\left|\widehat{Tu}(\xi')\right|^2 \leqslant \frac{1}{(2\pi)^2} \left(\int_{\mathbb{R}} (1 + |\xi|^2)^s \, |\hat{u}(\xi', \xi_n)|^2 \, d\xi_n \right) \left(\int_{\mathbb{R}} \frac{d\xi_n}{\left(1 + |\xi|^2\right)^s} \right),$$

where $\xi = (\xi', \xi_n)$.

c) By changing variables in the second integral above to $\xi_n = t\sqrt{1 + |\xi'|^2}$, show that there exists a constant $C(s)$ such that:

$$\|Tu\|_{H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})} \leqslant C(s) \, \|u\|_{H^s(\mathbb{R}^n)}.$$

d) Conclude that T extends to a bounded linear operator $T : H^s(\mathbb{R}^n) \to H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})$.

e) (*) Suppose $v \in \mathcal{S}(\mathbb{R}^{n-1})$ and let $\phi \in C_c^\infty(\mathbb{R})$ satisfy $\int_{\mathbb{R}} \phi(t)dt = \sqrt{2\pi}$. Define u through its Fourier transform by:

$$\hat{u}(\xi', \xi_n) = \frac{\hat{v}(\xi')}{\sqrt{1 + |\xi'|^2}} \phi\left(\frac{\xi_n}{\sqrt{1 + |\xi'|^2}} \right).$$

Show that there exists a constant $C > 0$ such that:

$$\|u\|_{H^s(\mathbb{R}^n)} \leqslant C \, \|v\|_{H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})}$$

and that $Tu = v$. Conclude that $T : H^s(\mathbb{R}^n) \to H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})$ is surjective.

Exercise 9. Suppose that $\Omega \subset \mathbb{R}^n$ is open and bounded. For $u \in H_0^1(\Omega)$, define the Dirichlet energy:

$$E[u] = \int_{\Omega} |Du|^2 \, dx.$$

a) Suppose that $(u_i)_{i=1}^\infty$ is a sequence with $u_i \in H_0^1(\Omega)$ such that $u_i \rightharpoonup u$. Show that $E[u] \leqslant \liminf_i E[u_i]$.

b) Consider the set
$$\mathcal{E}_1 = \{E[u] : u \in H^1_0(\Omega), \|u\|_{L^2} = 1\}$$
Let $\lambda_1 := \inf \mathcal{E}$. Show that there exists $w_1 \in H^1_0(\Omega)$ with $\|w_1\|_{L^2} = 1$ and $E[w_1] = \lambda_1$, and deduce $\lambda_1 > 0$.

c) Deduce that:
$$\lambda_1 \|u\|^2_{L^2} \leqslant \int_\Omega |Du|^2 \, dx$$
holds for all $u \in H^1_0(\Omega)$, with equality for $u = w_1$. This is *Poincaré's inequality*.

d) By considering $u = w_1 + t\phi$ for $t \in \mathbb{R}$, $\phi \in \mathscr{D}(\Omega)$, or otherwise, show that w_1 satisfies
$$-\Delta w_1 = \lambda_1 w_1,$$
where we understand this equation as holding in $\mathscr{D}'(\Omega)$.

e) (*) Suppose $\chi \in C^\infty_c(\Omega)$, and let $v = \chi w_1$. Show that v satisfies $-\Delta v + v = f$, where we understand the equation as holding in $\mathcal{S}'(\mathbb{R}^n)$, where $f \in L^2(\mathbb{R}^n)$. Deduce that $v \in H^2(\mathbb{R}^n)$. By iterating this argument, deduce that $w_1 \in H^1_0(\Omega) \cap C^\infty(\Omega)$.

f) (*) By considering
$$\mathcal{E}_2 = \{E[u] : u \in H^1_0(\Omega), \|u\|_{L^2} = 1, (u, w_1)_{L^2} = 0\},$$
or otherwise, show that there exists $\lambda_2 \geqslant \lambda_1$ and $w_2 \in H^1_0(\Omega) \cap C^\infty(\Omega)$ with $w_2 \neq w_1$, $\|w_2\|_{L^2} = 1$ solving
$$-\Delta w_2 = \lambda_2 w_2.$$