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Exercise 1. a) Show that $\mathcal{S}(\mathbb{R}^n)$ is a vector subspace of $\mathcal{E}(\mathbb{R}^n)$. Show that if $\{\phi_j\}_{j=1}^\infty$ is a sequence of rapidly decreasing functions which tends to zero in $\mathcal{S}(\mathbb{R}^n)$, then $\phi_j \to 0$ in $\mathcal{E}(\mathbb{R}^n)$.

b) Show that $\mathcal{D}(\mathbb{R}^n)$ is a vector subspace of $\mathcal{S}(\mathbb{R}^n)$. Show that if $\{\phi_j\}_{j=1}^\infty$ is a sequence of compactly supported functions which tends to zero in $\mathcal{D}(\mathbb{R}^n)$ then $\phi_j \to 0$ in $\mathcal{S}(\mathbb{R}^n)$.

c) Give an example of a sequence $\{\phi_j\}_{j=1}^\infty \subset C_c^\infty(\mathbb{R}^n)$ such that

i) $\phi_j \to 0$ in $\mathcal{S}(\mathbb{R}^n)$, but ϕ_j has no limit in $\mathcal{D}(\mathbb{R}^n)$.

ii) $\phi_j \to 0$ in $\mathcal{E}(\mathbb{R}^n)$, but ϕ_j has no limit in $\mathcal{S}(\mathbb{R}^n)$.

Exercise 2. For each $X \in \{\mathcal{D}(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n), \mathscr{E}(\mathbb{R}^n)\}$, suppose $\phi \in X$ and establish:

a) If $x_l \in \mathbb{R}^n$, $x_l \to 0$, then

$$\tau_{x_l}\phi \to \phi, \qquad \text{in } X \text{ as } l \to \infty,$$

where τ_x is the translation operator defined by $\tau_x\phi(y) := \phi(y - x)$.

b) If $h_l \in \mathbb{R}$, $h_l \to 0$, then

$$\Delta_i^{h_l}\phi \to D_i\phi, \qquad \text{in } X \text{ as } l \to \infty,$$

in X, where $\Delta_i^h\phi := h^{-1}\left[\tau_{-he_i}\phi - \phi\right]$ is the difference quotient.

Exercise 3. Suppose $u \in \mathcal{D}'(\mathbb{R})$ satisfies $Du = 0$. Show that u is a constant distribution, i.e. there exists $\lambda \in \mathbb{C}$ such that:

$$u[\phi] = \lambda \int_{\mathbb{R}} \phi(x)dx, \qquad \text{for all } \phi \in \mathcal{D}(\mathbb{R}).$$

(*) Extend the result to $\mathbb{R}^n$ for $n > 1$.
[Hint: Fix $\phi_0 \in \mathcal{D}(\mathbb{R})$ and show that any $\phi \in \mathcal{D}(\mathbb{R})$ may be written as $\phi(x) = \psi'(x) + c_\phi\phi_0(x)$ for some $\psi \in \mathcal{D}(\mathbb{R})$, $c_\phi \in \mathbb{C}$.]

Exercise 4. Let $X \in \{\mathcal{D}(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n), \mathscr{E}(\mathbb{R}^n)\}$. For $u \in X'$, $x \in \mathbb{R}^n$, define $\tau_x u$ by $\tau_x u[\phi] = u[\tau_{-x}\phi]$ for all $\phi \in X$, and let $\Delta_i^h u = h^{-1}\left[\tau_{-he_i}u - u\right]$. Show that $\Delta_i^h u \to D_i u$ as $h \to 0$ in the weak-$*$ topology of X'.

Exercise 5. Suppose $u \in \mathcal{D}'(\mathbb{R})$ satisfies $xu = 0$. Show that $u = c\delta_0$ for some $c \in \mathbb{C}$. Find the most general $u \in \mathcal{D}'(\mathbb{R})$ which satisfies $x^k u = 0$ for some $k \in \mathbb{N}$.

Exercise 6. Suppose $u : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ is a linear map. Show that u is continuous if and only if there exist $N, k \in \mathbb{N}$ and $C > 0$ such that:

$$|u[\phi]| \leqslant C \sup_{x\in\mathbb{R}^n; |\alpha|\leqslant k} \left|(1 + |x|)^N D^\alpha\phi(x)\right|, \qquad \text{for all } \phi \in \mathcal{S}(\mathbb{R}^n).$$

Exercise 7. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = e^x \cos(e^x)$. Determine whether the corresponding distribution belongs to $\mathcal{D}'(\mathbb{R}), \mathcal{S}'(\mathbb{R}), \mathscr{E}'(\mathbb{R})$, respectively.

Exercise 8. Suppose $u \in \mathcal{D}'(\mathbb{R}^n)$ is *positive*, i.e. $u[\phi] \geq 0$ for all $\phi \in \mathcal{D}(\mathbb{R}^n)$ with $\phi \geq 0$. Show that u has order 0, i.e., for each $K \subset \mathbb{R}^n$ compact, there is a constant C such that

$$|u[\phi]| \leq C \sup_{x\in K} |\phi(x)|, \qquad \text{for all } \phi \in C_c^\infty(K).$$

Exercise 9. Suppose $f \in L^1(\mathbb{R}^n)$, with $\operatorname{supp} f \subset B_R(0)$ for some $R > 0$. Show that $\hat{f} \in C^\infty(\mathbb{R}^n)$ and for any multi-index:

$$\sup_{\xi\in\mathbb{R}^n} \left|D^\alpha \hat{f}(\xi)\right| \leqslant R^{|\alpha|} \|f\|_{L^1}$$

Exercise 10. Recall that $L^\infty(\mathbb{R}) = L^1(\mathbb{R})'$. Consider the sequence $(f_n)_{n=1}^\infty$, where $f_n \in L^\infty(\mathbb{R})$ is given by $f_n(x) = \sin(nx)$. Show that $f_n \overset{*}{\rightharpoonup} 0$. Show that $f_n^2 \overset{*}{\rightharpoonup} g$ for some $g \in L^\infty(\mathbb{R})$ which you should find.

Exercise 11. Suppose $f \in \mathcal{S}(\mathbb{R}^n)$. By observing that

$$\|f\|_{L^2}^2 = \int_{\mathbb{R}^n} \frac{1}{n} (\operatorname{div} x)\, |f(x)|^2\, dx,$$

or otherwise, show that:

$$(2\pi)^{\frac{n}{2}} \|f\|_{L^2}^2 \leqslant \frac{2}{n} \big\| |x|\, f(x) \big\|_{L^2} \big\| |\xi|\, \hat{f}(\xi) \big\|_{L^2}$$

with equality if and only if $f(x) = a e^{-\lambda |x|^2}$ for some $a \in \mathbb{C}, \lambda > 0$. Deduce that if $x_0, \xi_0 \in \mathbb{R}^n$:

$$(2\pi)^{\frac{n}{2}} \|f\|_{L^2}^2 \leqslant \frac{2}{n} \big\| |x - x_0|\, f(x) \big\|_{L^2} \big\| |\xi - \xi_0|\, \hat{f}(\xi) \big\|_{L^2}.$$

Explain how this shows that a function $f \in L^2(\mathbb{R}^n)$ cannot be sharply localised in both physical and Fourier space simultaneously. This is the *uncertainty principle*.

Exercise 12. Let $f : \mathbb{R} \to \mathbb{R}$ be the sign function

$$f(x) = \begin{cases} -1 & x < 0 \\ 1 & x \geqslant 0 \end{cases}$$

and define $f_R(x) = f(x) \mathbb{1}_{[-R,R]}(x)$.

a) Sketch $f_R(x)$, and show that $T_{f_R} \to T_f$ in $\mathcal{S}'(\mathbb{R})$ as $R \to \infty$.

b) Compute $\hat{f}_R(\xi)$, and show that for $\phi \in \mathcal{S}(\mathbb{R})$:

$$T_{\hat{f}_R}[\phi] = -2i \int_0^\infty \frac{\phi(x) - \phi(-x)}{x} dx + 2i \int_0^\infty \left(\frac{\phi(x) - \phi(-x)}{x} \right) \cos Rx dx.$$

Deduce $\widehat{T_f} = -2i P.V. \left(\frac{1}{x}\right)$, where we define the distribution $P.V. \left(\frac{1}{x}\right)$ by:

$$P.V. \left(\frac{1}{x}\right)[\phi] = \lim_{\epsilon \to 0} \int_{\mathbb{R} \setminus (-\epsilon, \epsilon)} \frac{\phi(x)}{x} dx, \qquad \phi \in \mathcal{S}(\mathbb{R}).$$

c) Write down $\widehat{T_H}$, where H is the Heaviside function:

$$H(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geqslant 0 \end{cases}$$

By considering $e^{-\epsilon x} H(x)$, or otherwise, find an expression for the distribution u which acts on $\phi \in \mathcal{S}(\mathbb{R})$ by:

$$u[\phi] := \lim_{\epsilon \to 0^+} \int_{\mathbb{R}} \frac{\phi(x)}{x + i\epsilon} dx.$$

Exercise 13. Suppose $\phi \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^m)$. For each $y \in \mathbb{R}^m$ let $\phi_y : \mathbb{R}^n \to \mathbb{C}$ be given by $\phi_y(x) = \phi(x, y)$. Let $u \in \mathcal{D}'(\mathbb{R}^n)$.

a) Show that $\psi : y \mapsto u[\phi_y]$ is smooth and find an expression for $D^\alpha \psi$. Deduce that

$$\int_{\mathbb{R}^m} \psi(y) dy = u[\Psi], \quad \text{where} \quad \Psi(x) = \int_{\mathbb{R}^m} \phi(x, y) dy.$$

b) Show that there exists a sequence of smooth functions $f_n \in C_c^\infty(\mathbb{R}^n)$ such that $T_{f_n} \to u$ in the weak-$*$ topology of $\mathcal{D}'(\mathbb{R}^n)$.