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Exercise 1. Let X be a normed space. Show that X' equipped with its norm forms a Banach space. If $\overline{X}$ is the completion of X with respect to the metric induced by its norm, show that $X' = \overline{X}'$.

Exercise 2. Suppose X is a Banach space. Show that if $\Lambda \in X'$ with $\Lambda \neq 0$ then Λ is an open mapping (i.e. $\Lambda(U)$ is open whenever $U \subset X$ is open).

Exercise 3. Let $u : L^p(\mathbb{R}^n; \mathbb{R}) \to \mathbb{R}$ be a bounded, linear functional.

a) For $f \in L^p(\mathbb{R}^n; \mathbb{R})$, $f \geqslant 0$, define

$$\tilde{u}(f) = \sup\{u(g) : g \in L^p(\mathbb{R}^n; \mathbb{R}), 0 \leqslant g \leqslant f\}.$$

Show that $0 \leqslant \tilde{u}(f)$ and $u(f) \leqslant \tilde{u}(f) \leqslant \|u\|_{L^{p'}} \|f\|_{L^p}$, and establish

$$\tilde{u}(f + ag) = \tilde{u}(f) + a\tilde{u}(g)$$

for all $f, g \in L^p(\mathbb{R}^n; \mathbb{R})$ with $f, g \geqslant 0$ and $a \in \mathbb{R}$, $a > 0$.

b) For $f \in L^p(\mathbb{R}^n; \mathbb{R})$, define $w(f) = \tilde{u}(f^+) - \tilde{u}(f^-)$, where $f^+(x) = \max\{0, f(x)\}$, $f^-(x) = \max\{0, -f(x)\}$. Show that w is linear and bounded, and that w and $w - u$ are positive.

c) Deduce that $u = u_+ - u_-$, where $u_\pm$ are bounded, positive, linear functionals.

Exercise 4. Suppose X is a normed space, and $V \subset X$ is a closed proper subspace of X and let $0 < \alpha < 1$. Show that there exists $x \in X$ with $\|x\| = 1$ such that $\|x - y\| \geqslant \alpha$ for all $y \in V$. Deduce that the Bolzano–Weiserstrass theorem does not hold if X is an infinite dimensional Banach space.
[The first result above is known as Riesz' Lemma]

Exercise 5. Let $\mathscr{P}$ be a separating family of seminorms on a vector space X. Show that a sequence $(x_k)_{k=1}^\infty$ with $x_k \in X$ converges to $x \in X$ in the topology $\tau_{\mathscr{P}}$ if and only if $p(x_k - x) \to 0$ for all $p \in \mathscr{P}$.

Exercise 6. Suppose that X is a Banach space, and let $(\Lambda_k)_{k=1}^\infty$ be a sequence with $\Lambda_k \in X'$. Show that:

$$\Lambda_k \to \Lambda \quad \Longrightarrow \quad \Lambda_k \rightharpoonup \Lambda \quad \Longrightarrow \quad \Lambda_k \overset{*}{\rightharpoonup} \Lambda.$$

Exercise 7. For a bounded measurable set $E \subset \mathbb{R}^n$ of positive measure, and any $f \in L^1_{loc.}(\mathbb{R}^n)$, define the mean of f on E to be:

$$⨍_E f(x)dx = \frac{1}{|E|} \int_E f(x)dx.$$

Suppose $1 < p < \infty$ and let $(f_j)_{j=1}^\infty$ be a bounded sequence in $L^p(\mathbb{R}^n)$. Show that $f_j \rightharpoonup f$ for some $f \in L^p(\mathbb{R}^n)$ if and only if

$$⨍_E f_j(x)dx \to ⨍_E f(x)dx$$

for all bounded measurable sets $E \subset \mathbb{R}^n$ of positive measure.

Exercise 8. Suppose $(H, (\cdot, \cdot))$ is an infinite dimensional Hilbert space and let $(x_i)_{i=1}^\infty$ be a sequence with $x_i \in H$.

i) Show that $x_i \rightharpoonup x$ if and only if $(y, x_i) \to (y, x)$ for all $y \in H$.

ii) Show there exists a sequence such that $x_i \rightharpoonup 0$, but $x_i \not\to 0$.

iii) Suppose $x_i \rightharpoonup x$. Show that

$$\|x\| \leqslant \liminf_{i\to\infty} \|x_i\| ,$$

and $\|x_i\| \to \|x\|$ iff $x_i \to x$.

Exercise 9. Construct a bounded sequence $(f_i)_{i=1}^{\infty}$ of functions $f_i \in L^1(\mathbb{R})$ such that no subsequence is weakly convergent.

Exercise 10. Let X be a Banach space and suppose $A \subset X$ is a convex neighbourhood of 0. For $x \in X$ define $\mu_A(x) = \inf\{t > 0 : t^{-1}x \in A\}$. Show that μ_A is sublinear and satisfies $\mu_A(x) \leqslant k\|x\|$ for some $k > 0$. Show further that $\mu_A(y) < 1$ for $y \in A$, A open. [μ_A *is called the Minkowski functional of* A]

Exercise 11. Let $\{x_1, \dots x_n\}$ be a set of linearly independent elements of a Banach space X. Let $a_1, \dots, a_n \in \mathbb{C}$. Show that there exists $\Lambda \in X'$ such that $\Lambda(x_i) = a_i$, for $i = 1, \dots, n$.

Exercise 12. Let M be a vector subspace of the Banach space X, and suppose that $K \subset X$ is open, convex and disjoint from M. Show that there exists a co-dimension one subspace $N \subset X$ which contains M and is disjoint from K. [*This is Mazur's theorem.*]

Exercise 13. Let X be a reflexive Banach space, and suppose $Y \subset X$ is a closed subspace. Show that Y is reflexive.