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Exercise 1. Suppose $f, g : E \to \mathbb{C}$ are measurable functions on some measure space $(E, \mathcal{E}, \mu)$. Show that:

a) $\|fg\|_{L^r} \leqslant \|f\|_{L^p} \|g\|_{L^q}$ where $1 \leqslant p, q, r \leqslant \infty$ satisfy $p^{-1} + q^{-1} = r^{-1}$
[You may wish to first establish the special case $r = 1$.]

b) $\|f + g\|_{L^p} \leqslant \|f\|_{L^p} + \|g\|_{L^p}$ for $1 \leqslant p \leqslant \infty$.

Exercise 2. a) Suppose that $\mu(E) < \infty$. Show that if $f \in L^p(E, \mu)$, then $f \in L^q(E, \mu)$ for any $1 \leqslant q \leqslant p$, with

$$\|f\|_{L^q} \leqslant \mu(E)^{\frac{p-q}{qp}} \|f\|_{L^p} .$$

b) Suppose that $f \in L^{p_0}(E, \mu) \cap L^{p_1}(E, \mu)$ with $p_0 < p_1 \leqslant \infty$. For $0 \leqslant \theta \leqslant 1$, define p_θ by

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

Show that $f \in L^{p_\theta}(E, \mu)$ with

$$\|f\|_{L^{p_\theta}} \leqslant \|f\|_{L^{p_0}}^{1-\theta} \|f\|_{L^{p_1}}^{\theta} .$$

c) Show that for $p_1 \neq p_2$ we have $L^{p_1}(\mathbb{R}^n) \not\subset L^{p_2}(\mathbb{R}^n)$. For which p_1, p_2 do we have $L^{p_1}_{loc.}(\mathbb{R}^n) \subset L^{p_2}_{loc.}(\mathbb{R}^n)$?

Exercise 3. Let $\mathcal{R}_\mathbb{Q}$ be the set of rectangles of the form $(a_1, b_1] \times \cdots \times (a_n, b_n]$ with $a_i, b_i \in \mathbb{Q}$, and let $S_\mathbb{Q}$ be the set of functions of the form

$$s(x) = \sum_{k=1}^{N} (\alpha_k + i\beta_k) \mathbb{1}_{R_k}$$

for $R_k \in \mathcal{R}_\mathbb{Q}$ and $\alpha_k, \beta_k \in \mathbb{Q}$. For $1 \leqslant p < \infty$ show that $S_\mathbb{Q}$ is dense in $L^p(\mathbb{R}^n)$ and deduce that $L^p(\mathbb{R}^n)$ is separable. Show that $L^\infty(\mathbb{R}^n)$ is not separable.
[Hint: for the last part exhibit an uncountable subset $X \subset L^\infty(\mathbb{R}^n)$ such that $\|f - g\|_{L^\infty(\mathbb{R}^n)} \geqslant 1$ for any $f, g \in X$, $f \neq g$] .

Exercise 4. a) Suppose $1 \leqslant p \leqslant \infty$ and let q satisfy $p^{-1} + q^{-1} = 1$. Show that for a measurable function $f : \mathbb{R}^n \to \mathbb{C}$:

$$\|f\|_{L^p} = \sup \left\{ \int_{\mathbb{R}^n} |f(x)g(x)|\, dx : g \in L^q(\mathbb{R}^n), \|g\|_{L^q} \leqslant 1 \right\} .$$

b) Now suppose $p < \infty$ and assume $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ is integrable. Set $G(y) = \int_{\mathbb{R}^n} F(x, y) dx$. Show that if $\|g\|_{L^q} \leqslant 1$ then

$$\int_{\mathbb{R}^n} |G(y)g(y)|\, dy \leqslant \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} |F(x, y)|^p\, dy \right]^{\frac{1}{p}} dx.$$

Deduce Minkowski's integral inequality

$$\left[\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} F(x, y) dx \right|^p dy \right]^{\frac{1}{p}} \leqslant \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} |F(x, y)|^p\, dy \right]^{\frac{1}{p}} dx.$$

Exercise 5. Let $I = (0, 1)$ and $1 \leq p < \infty$. Exhibit a sequence $(f_j)_{j=1}^\infty$ with $f_j \in L^p(I)$ such that $f_j \to 0$ in $L^p(I)$, but $f_j(x)$ does not converge for any x. Does such a sequence exist if $p = \infty$?

Exercise 6. Suppose $1 \leqslant p < \infty$.

a) Suppose $f \in L^p(\mathbb{R}^n)$. Show that

$$|\{x : |f(x)| > \lambda\}| \leqslant \frac{\|f\|_{L^p}^p}{\lambda^p}.$$

This is known as Tchebychev's inequality, the $p = 1$ *case is Markov's inequality.*

b) We say that a measurable $f : \mathbb{R}^n \to \mathbb{C}$ is in *weak-*$L^p(\mathbb{R}^n)$, written $f \in L^{p,w}(\mathbb{R}^n)$ if there exists a constant C such that

$$|\{x : |f(x)| > \lambda\}| \leqslant \frac{C^p}{\lambda^p}.$$

Show that $L^p(\mathbb{R}^n) \subset L^{p,w}(\mathbb{R}^n)$, and that the inclusion is proper.

Exercise 7. Suppose that $f \in L^r(\mathbb{R}^n)$ for some $1 \leqslant r < \infty$. Show that $\|f\|_{L^\infty} = \lim_{p\to\infty} \|f\|_{L^p}$.
[Hint: you may find the estimates in Exercises 2 b), 6 a) useful.]

Exercise 8. a) Let $B_1, \ldots, B_N$ be a finite collection of open balls in $\mathbb{R}^n$. Show that there exists a subcollection $B_{i_1}, \ldots, B_{i_k}$ of *disjoint* balls such that

$$\bigcup_{i=1}^N B_i \subset \bigcup_{j=1}^k (3B_{i_j}),$$

where $3B$ is the ball with the same centre as B but three times the radius. Deduce

$$\left|\bigcup_{i=1}^N B_i\right| \leqslant 3^n \sum_{j=1}^k \left|B_{i_j}\right|.$$

b) (*) Suppose $\{B_j : j \in J\}$ is an arbitrary collection of balls in $\mathbb{R}^n$ such that each ball has radius at most R. Show that there exists a countable subcollection $\{B_j : j \in J'\}$, $J' \subset J$ of disjoint balls such that

$$\bigcup_{i\in J} B_i \subset \bigcup_{i\in J'} (5B_i).$$

These are Wiener and Vitali's covering Lemmas, respectively.

Exercise 9. Suppose $f : \mathbb{R} \to \mathbb{C}$ is integrable and let $F(x) = \int_{-\infty}^x f(t)dt$. Show that F is differentiable with $F'(x) = f(x)$ at each Lebesgue point $x \in \mathbb{R}$. Deduce that F is differentiable almost everywhere.

Exercise 10. Suppose $\phi \in L^\infty(\mathbb{R}^n)$ satisfies $\phi \geqslant 0$, $\operatorname{supp} \phi \subset B_1(0)$, and $\int_{\mathbb{R}^n} \phi\, dx = 1$. Set $\phi_\epsilon(x) = \epsilon^{-n}\phi(\epsilon^{-1}x)$. Show that if $f \in L^1(\mathbb{R}^n)$, and x is a Lebesgue point of f,

$$\phi_\epsilon \star f(x) \to f(x), \qquad \text{as } \epsilon \to 0.$$

Exercise 11. Let $S = \{\psi_{n,k}\}_{n,k\in\mathbb{Z}}$ be the Haar system, as defined in lectures.

a) Show that

$$\int_{\mathbb{R}} \psi_{n_1,k_1}(x)\psi_{n_2,k_2}(x)dx = \delta_{n_1 n_2}\delta_{k_1 k_2}.$$

b) Show that $\mathbb{1}_I \in \overline{\operatorname{Span} S}$ for any finite interval I, where the closure is understood with respect to the L^2 norm.

c) Deduce that S is an orthonormal basis for $L^2(\mathbb{R})$.

Exercise 12. (*) Suppose $(E, \mathcal{E})$ is a measurable space, with finite measures μ, ν. Show that ν may be uniquely written as $\nu = \nu_a + \nu_s$, for measures ν_a, ν_s such that $\nu_s \perp \mu$ and $\nu_a \ll \mu$.
[Hint: Return to the proof of the Radon–Nikodym theorem, but drop the assumption that $\nu \ll \mu$*]*