Example Sheet 2	Analysis of Functions
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Exercise 2.1. Let X be a normed space. Show that X' equipped with its norm forms a Banach space. If \overline{X} is the completion of X with respect to the metric induced by its norm, show that $X' = \overline{X}'$.

Exercise 2.2. Suppose X is a Banach space. Show that if $\Lambda \in X'$ with $\Lambda \neq 0$ then Λ is an open mapping (i.e. $\Lambda(U)$ is open whenever $U \subset X$ is open).

Exercise 2.3. Let $u: L^p(\mathbb{R}^n; \mathbb{R}) \to \mathbb{R}$ be a bounded, linear functional.

a) For $f \in L^p(\mathbb{R}^n; \mathbb{R}), f \ge 0$, define

 $\tilde{u}(f) = \sup\{u(g) : g \in L^p(\mathbb{R}^n; \mathbb{R}), 0 \leqslant g \leqslant f\}.$

Show that $0 \leq \tilde{u}(f)$ and $u(f) \leq \tilde{u}(f) \leq ||u||_{L^{p'}} ||f||_{L^p}$, and establish

$$\tilde{u}(f + ag) = \tilde{u}(f) + a\tilde{u}(g)$$

for all $f, g \in L^p(\mathbb{R}^n; \mathbb{R})$ with $f, g \ge 0$ and $a \in \mathbb{R}, a > 0$.

- b) For $f \in L^p(\mathbb{R}^n; \mathbb{R})$, define $w(f) = \tilde{u}(f^+) \tilde{u}(f^-)$, where $f^+(x) = \max\{0, f(x)\}$, $f^-(x) = \max\{0, -f(x)\}$. Show that w is linear and bounded, and that w and w u are positive.
- c) Deduce that $u = u_{+} u_{-}$, where u_{\pm} are bounded, positive, linear functionals.

Exercise 2.4. Suppose X is a normed space, and $V \subset X$ is a closed proper subspace of X and let $0 < \alpha < 1$. Show that there exists $x \in X$ with ||x|| = 1 such that $||x - y|| \ge \alpha$ for all $y \in V$. Deduce that the Bolzano–Weiserstrass theorem does not hold if X is an infinite dimensional Banach space.

[The first result above is known as Riesz' Lemma]

Exercise 2.5. Let \mathscr{P} be a separating family of seminorms on a vector space X. Show that a sequence $(x_k)_{k=1}^{\infty}$ with $x_k \in X$ converges to $x \in X$ in the topology $\tau_{\mathscr{P}}$ if and only if $p(x_k - x) \to 0$ for all $p \in \mathscr{P}$.

Exercise 2.6. Suppose that X is a Banach space, and let $(\Lambda_k)_{k=1}^{\infty}$ be a sequence with $\Lambda_k \in X'$. Show that:

$$\Lambda_k \to \Lambda \implies \Lambda_k \rightharpoonup \Lambda \implies \Lambda_k \stackrel{*}{\rightharpoonup} \Lambda.$$

(*) Show the stronger statement that $\tau_{w*} \subset \tau_w \subset \tau_s$, where $\tau_{w*}, \tau_w, \tau_s$ are the weak-*, weak and strong topologies on X' respectively.

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Exercise 2.7. For a bounded measurable set $E \subset \mathbb{R}^n$ of positive measure, and any $f \in L^1_{loc.}(\mathbb{R}^n)$, define the mean of f on E to be:

$$\int_{E} f(x)dx = \frac{1}{|E|} \int_{E} f(x)dx.$$

Suppose $1 and let <math>(f_j)_{j=1}^{\infty}$ be a bounded sequence in $L^p(\mathbb{R}^n)$. Show that $f_j \rightharpoonup f$ for some $f \in L^p(\mathbb{R}^n)$ if and only if

$$\oint_E f_j(x)dx \to \oint_E f(x)dx$$

for all bounded measurable sets $E \subset \mathbb{R}^n$ of positive measure.

Exercise 2.8. Suppose $(H, (\cdot, \cdot))$ is an infinite dimensional Hilbert space and let $(x_i)_{i=1}^{\infty}$ be a sequence with $x_i \in H$.

- i) Show that $x_i \rightarrow x$ if and only if $(y, x_i) \rightarrow (y, x)$ for all $y \in H$.
- ii) Show there exists a sequence such that $x_i \rightarrow 0$, but $x_i \not\rightarrow 0$.
- iii) Suppose $x_i \rightharpoonup x$. Show that

$$||x|| \leq \liminf_{i \to \infty} ||x_i|| ,$$

and $||x_i|| \to ||x||$ iff $x_i \to x$.

Exercise 2.9. Construct a bounded sequence $(f_i)_{i=1}^{\infty}$ of functions $f_i \in L^1(\mathbb{R})$ such that no subsequence is weakly convergent.

Exercise 2.10. Let X be a Banach space and suppose $A \subset X$ is a convex neighbourhood of 0. For $x \in X$ define $\mu_A(x) = \inf\{t > 0 : t^{-1}x \in A\}$. Show that μ_A is sublinear and satisfies $\mu_A(x) \leq k ||x||$ for some k > 0. Show further that $\mu_A(y) < 1$ for $y \in A$. μ_A is called the Minkowski functional of A

Exercise 2.11. Let $\{x_1, \ldots, x_n\}$ be a set of linearly independent elements of a Banach space X. Let $a_1, \ldots, a_n \in \mathbb{C}$. Show that there exists $\Lambda \in X'$ such that $\Lambda(x_i) = a_i$, for $i = 1, \ldots, n$.

Exercise 2.12. Let M be a vector subspace of the Banach space X, and suppose that $K \subset X$ is open, convex and disjoint from M. Show that there exists a co-dimension one subspace $N \subset X$ which contains M and is disjoint from K. This is Mazur's theorem.

Exercise 2.13. let X be a reflexive Banach space, and suppose $Y \subset X$ is a closed subspace. Show that Y is reflexive.