Exercise 3.1. Suppose $u : \mathscr{S} \to \mathbb{C}$ is a linear map. Show that u is continuous if and only if there exist $N, k \in \mathbb{N}$ and $C > 0$ such that:

$$|u[\phi]| \leqslant C \sup_{x\in\mathbb{R}^n;|\alpha|\leqslant k} \left|(1+|x|)^N D^\alpha \phi(x)\right|, \qquad \text{for all } \phi \in \mathscr{S}.$$

Exercise 3.2. Suppose $u \in \mathscr{D}'(\mathbb{R}^n)$ is *positive*, i.e. $u[\phi] \geq 0$ for all $\phi \in \mathscr{D}(\mathbb{R}^n)$ with $\phi \geq 0$. Show that u has order 0. (*) Deduce that $u[\phi] = \int_{\mathbb{R}^n} \phi d\mu$ for some regular measure μ.

Exercise 3.3. Suppose $f \in L^1(\mathbb{R}^n)$, with $\operatorname{supp} f \subset B_R(0)$ for some $R > 0$.

a) Show that $\hat{f} \in C^\infty(\mathbb{R}^n)$ and for any multi-index:

$$\sup_{\xi\in\mathbb{R}^n} \left|D^\alpha \hat{f}(\xi)\right| \leqslant R^{|\alpha|} \|f\|_{L^1}$$

b) (*)Show that $\hat{f}$ is real analytic, with an infinite radius of convergence, i.e.:

$$\hat{f}(\xi) = \sum_\alpha D^\alpha \hat{f}(0) \frac{\xi^\alpha}{\alpha!}$$

holds for all $\xi \in \mathbb{R}^n$.Deduce that if $\hat{f}(\xi)$ vanishes on an open set, it must vanish everywhere.

You may assume the following form of Taylor's theorem. Suppose $g \in C^{k+1}(\overline{B_r(0)})$. Then for $x \in B_r(0)$:

$$g(x) = \sum_{|\alpha|\leqslant k} D^\alpha g(0) \frac{\xi^\alpha}{\alpha!} + \sum_{|\beta|=k+1} R_\beta(x) x^\beta$$

where the remainder $R_\beta(x)$ satisfies the following estimate in $B_r(0)$:

$$|R_\beta(x)| \leqslant \frac{1}{\beta!} \max_{|\alpha|=|\beta|} \max_{y\in\overline{B_r(0)}} |D^\alpha g(y)|.$$

Exercise 3.4. Recall that $L^\infty(\mathbb{R}) = L^1(\mathbb{R})'$. Consider the sequence $(f_n)_{n=1}^\infty$, where $f_n \in L^\infty(\mathbb{R})$ is given by $f_n(x) = \sin(nx)$. Show that $f_n \overset{*}{\rightharpoonup} 0$. Show that $f_n^2 \overset{*}{\rightharpoonup} g$ for some $g \in L^\infty(\mathbb{R})$ which you should find.

Exercise 3.5. Suppose $f \in \mathscr{S}(\mathbb{R}^n)$. By observing that

$$\|f\|_{L^2}^2 = \int_{\mathbb{R}^n} \frac{1}{n} (\operatorname{div} x) |f(x)|^2 \, dx,$$

Please send any corrections to c.m.warnick@maths.cam.ac.uk or otherwise, show that:

$$(2\pi)^{\frac{n}{2}} \|f\|_{L^2}^2 \leqslant \frac{2}{n} \big\| |x|\, f(x) \big\|_{L^2} \big\| \,|\xi|\, \hat{f}(\xi) \big\|_{L^2}$$

with equality if and only if $f(x) = a e^{-\lambda |x|^2}$ for some $a \in \mathbb{C}, \lambda > 0$. Deduce that if $x_0, \xi_0 \in \mathbb{R}^n$:

$$(2\pi)^{\frac{n}{2}} \|f\|_{L^2}^2 \leqslant \frac{2}{n} \big\| |x - x_0|\, f(x) \big\|_{L^2} \big\| \,|\xi - \xi_0|\, \hat{f}(\xi) \big\|_{L^2}.$$

Explain how this shows that a function $f \in L^2(\mathbb{R}^n)$ cannot be sharply localised in both physical and Fourier space simultaneously. This is the *uncertainty principle*.

Exercise 3.6. Let $f : \mathbb{R} \to \mathbb{R}$ be the sign function

$$f(x) = \begin{cases} -1 & x < 0 \\ 1 & x \geqslant 0 \end{cases}$$

and define $f_R(x) = f(x) \mathbb{1}_{[-R,R]}(x)$.

a) Sketch $f_R(x)$, and show that $T_{f_R} \to T_f$ in $\mathscr{S}'(\mathbb{R})$ as $R \to \infty$.

b) Compute $\hat{f}_R(\xi)$, and show that for $\phi \in \mathscr{S}(\mathbb{R})$:

$$T_{\hat{f}_R}[\phi] = -2i \int_0^\infty \frac{\phi(x) - \phi(-x)}{x} dx + 2i \int_0^\infty \left(\frac{\phi(x) - \phi(-x)}{x} \right) \cos Rx dx.$$

Deduce $\widehat{T_f} = -2i P.V. \left(\frac{1}{x}\right)$, where we define the distribution $P.V. \left(\frac{1}{x}\right)$ by:

$$P.V. \left(\frac{1}{x} \right) [\phi] = \lim_{\epsilon \to 0} \int_{\mathbb{R} \setminus (-\epsilon, \epsilon)} \frac{\phi(x)}{x} dx, \qquad \phi \in \mathscr{S}(\mathbb{R}).$$

c) Write down $\widehat{T_H}$, where H is the Heaviside function:

$$H(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geqslant 0 \end{cases}$$

By considering $e^{-\epsilon x} H(x)$, or otherwise, find an expression for the distribution u which acts on $\phi \in \mathscr{S}(\mathbb{R})$ by:

$$u[\phi] := \lim_{\epsilon \to 0^+} \int_{\mathbb{R}} \frac{\phi(x)}{x + i\epsilon} dx.$$

Exercise 3.7. Let $s \in \mathbb{R}$.

a) Show that $\mathscr{S}$ is a dense subset of $H^s(\mathbb{R}^n)$.

b) Find a condition on s such that $\delta_x \in H^s(\mathbb{R}^n)$.

c) Show that $H^t(\mathbb{R}^n)$ is continuously embedded in $H^s(\mathbb{R}^n)$ for $t < s$.

d) Show that the derivative D^α is a bounded linear map from $H^{s+k}(\mathbb{R}^n)$ into $H^s(\mathbb{R}^n)$, where $k = |\alpha|$.

e) (*) Show that the pairing $\langle , \rangle : H^{-s}(\mathbb{R}^n) \times H^s(\mathbb{R}^n) \to \mathbb{C}$, which acts on $f \in H^{-s}(\mathbb{R}^n), g \in H^s(\mathbb{R}^n)$ by

$$\langle f, g\rangle = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \hat{f}(\xi)\hat{g}(\xi)d\xi$$

is well defined, and show that the map $g \mapsto \langle f, g\rangle$ is a bounded linear operator on $H^s(\mathbb{R}^n)$. Deduce that $H^s(\mathbb{R}^n)'$ may be identified with $H^{-s}(\mathbb{R}^n)$. How does this relate to your answer to part b)?

Exercise 3.8. a) Suppose $s = \frac{n}{2} + \gamma$ for some $0 < \gamma < 1$. Show that there exists a constant $C_{n,\gamma} > 0$ such that for all $x, y \in \mathbb{R}^n$:

$$\int_{\mathbb{R}^n} \frac{\left|e^{ix\cdot\xi} - e^{iy\cdot\xi}\right|^2}{|\xi|^{2s}} d\xi \leqslant C_{n,\gamma} |x - y|^{2\gamma}$$

b) Show that if $s = \frac{n}{2} + k + \gamma$ for some $k \in \mathbb{Z}_{\geqslant 0}$, $0 < \gamma < 1$, then

$$H^s(\mathbb{R}^n) \subset C^{k,\gamma}(\mathbb{R}^n).$$

Exercise 3.9. Fix $s \in \mathbb{R}$, and suppose that $f \in H^s(\mathbb{R}^n)$.

a) Show that there exists a unique $u \in H^{s+4}(\mathbb{R}^n)$ which solves:

$$\Delta^2 u + u = f.$$

b) Show further that there exists $C > 0$ such that $\|u\|_{H^{s+4}} \leqslant C\|f\|_{H^s}$.

c) For what values of s does the equation hold in the sense of classical derivatives (possibly after redefining u, f on a set of measure zero)?

Exercise 3.10. Assume $s > \frac{1}{2}$ and suppose $u \in \mathscr{S}(\mathbb{R}^n)$. Define $Tu \in \mathscr{S}(\mathbb{R}^{n-1})$ by:

$$Tu(x') = u(x', 0), \qquad x' \in \mathbb{R}^{n-1}.$$

a) Show that if $\xi' \in \mathbb{R}^{n-1}$:

$$\widehat{Tu}(\xi') = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \hat{u}(\xi', \xi_n)d\xi_n.$$

b) Deduce that:

$$\left|\widehat{Tu}(\xi')\right|^2 \leqslant \frac{1}{2\pi}\left(\int_{\mathbb{R}} (1 + |\xi|^2)^s \left|\hat{u}(\xi', \xi_n)\right|^2 d\xi_n\right)\left(\int_{\mathbb{R}} \frac{d\xi_n}{\left(1 + |\xi|^2\right)^s}\right),$$

where $\xi = (\xi', \xi_n)$.

c) By changing variables in the second integral above to $\xi_n = t\sqrt{1+|\xi'|^2}$, conclude that there exists a constant $C(s)$ such that:

$$\|Tu\|_{\hat{H}^{s-\frac{1}{2}}(\mathbb{R}^{n-1})} \leqslant C(s) \|u\|_{\hat{H}^s(\mathbb{R}^n)} .$$

d) Conclude that T extends to a bounded linear operator $T : H^s(\mathbb{R}^n) \to H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})$.

e) (*) Suppose $v \in \mathscr{S}(\mathbb{R}^{n-1})$ and let $\phi \in C^\infty_c(\mathbb{R})$ satisfy $\int_{\mathbb{R}} \phi(t)dt = \sqrt{2\pi}$. Define u through its Fourier transform by:

$$\hat{u}(\xi', \xi_n) = \frac{\hat{v}(\xi')}{\sqrt{1+|\xi'|^2}}\phi\left(\frac{\xi_n}{\sqrt{1+|\xi'|^2}}\right).$$

Show that there exists a constant $C > 0$ such that:

$$\|u\|_{\hat{H}^s(\mathbb{R}^n)} \leqslant C \|v\|_{\hat{H}^{s-\frac{1}{2}}(\mathbb{R}^{n-1})}$$

and that $Tu = v$. Conclude that $T : \hat{H}^s(\mathbb{R}^n) \to \hat{H}^{s-\frac{1}{2}}(\mathbb{R}^{n-1})$ is surjective.

The following are longer-form questions designed to show how some of the concepts of the course can be applied to solve interesting problems. They are optional, but you may wish to attempt one or two that you find interesting.

Exercise 3.11. Suppose that $\Omega \subset \mathbb{R}^n$ is open and bounded. For $u \in H^1_0(\Omega)$, define the Dirichlet energy:

$$E[u] = \int_\Omega |Du|^2 \, dx.$$

a) Suppose that $(u_i)_{i=1}^\infty$ is a sequence with $u_i \in H^1_0(\Omega)$ such that $u_i \rightharpoonup u$. Show that $E[u] \leqslant \liminf_i E[u_i]$.

b) Consider the set

$$\mathcal{E}_1 = \{E[u] : u \in H^1_0(\Omega), \|u\|_{L^2} = 1\}$$

Let $\lambda_1 := \inf \mathcal{E}$. Show that there exists $w_1 \in H^1_0(\Omega)$ with $\|w_1\|_{L^2} = 1$ and $E[w_1] = \lambda_1$, and deduce $\lambda_1 > 0$.

c) Deduce that:

$$\lambda_1 \|u\|^2_{L^2} \leqslant \int_\Omega |Du|^2 \, dx$$

holds for all $u \in H^1_0(\Omega)$, with equality for $u = w_1$. This is *Poincaré's inequality.*

d) By considering $u = w_1 + t\phi$ for $t \in \mathbb{R}$, $\phi \in \mathscr{D}(\Omega)$, or otherwise, show that w_1 satisfies

$$-\Delta w_1 = \lambda_1 w_1,$$

where we understand this equation as holding in $\mathscr{D}'(\Omega)$.

e) (*) Suppose $\chi \in C_c^\infty(\Omega)$, and let $v = \chi w_1$. Show that v satisfies $-\Delta v + v = f$, where we understand the equation as holding in $\mathscr{S}'(\mathbb{R}^n)$, where $f \in L^2(\mathbb{R}^n)$. Deduce that $v \in H^2(\mathbb{R}^n)$. By iterating this argument, deduce that $w_1 \in H_0^1(\Omega) \cap C^\infty(\Omega)$.

f) (*) By considering

$$\mathcal{E}_2 = \{E[u] : u \in H_0^1(\Omega), \|u\|_{L^2} = 1, (u, w_1)_{L^2} = 0\},$$

or otherwise, show that there exists $\lambda_2 \geqslant \lambda_1$ and $w_2 \in H_0^1(\Omega) \cap C^\infty(\Omega)$ with $w_2 \neq w_1$, $\|w_2\|_{L^2} = 1$ solving

$$-\Delta w_2 = \lambda_2 w_2.$$

Exercise 3.12. Let H be the completion of $\mathscr{S}(\mathbb{R}^n)$ with respect to the norm

$$\|u\|_H := \left(\int_{\mathbb{R}^n} \left(|Du|^2 + |x|^2 |u|^2 \right) dx \right)^{\frac{1}{2}}$$

a) Show that H is a Hilbert space with the inner product:

$$(u, v)_H := \int_{\mathbb{R}^n} \left(\overline{Du} \cdot Dv + |x|^2 \overline{u} v \right) dx,$$

and show that if $u \in H, \chi \in C_c^\infty(B_R(0))$, then $\chi u \in H_0^1(B_R(0))$, with $\|\chi u\|_{H^1} \leqslant C_{R,\chi} \|u\|_H$ for some constant $C_{R,\chi} > 0$.

b) Show that H embeds compactly into $L^2(\mathbb{R}^n)$, that is $H \subset L^2(\mathbb{R}^n)$ and if $(u_n)_{n=1}^\infty$ is a bounded sequence in H then it admits a subsequence which converges in $L^2(\mathbb{R}^n)$. *[Hint: take a subsequence converging weakly in both H and $L^2(\mathbb{R}^n)$, and write $u_n = u_n \chi_R + u_n(1 - \chi_R)$, where $\chi_R \in C_c^\infty(B_R(0))$ satisfies $\chi_R(x) = 1$ for $|x| < R - 1$, where R is to be chosen.]*

c) If $f \in L^2(\mathbb{R}^n)$, we say that $u \in H$ is a weak solution of:

$$-\Delta u + |x|^2 u = f \qquad (\dagger)$$

if

$$(u, v)_H = (f, v)_{L^2} \text{ for all } v \in H. \qquad (\diamond)$$

Show that if $u, f \in \mathscr{S}(\mathbb{R}^n)$ solve $(\dagger)$, then u satisfies $(\diamond)$. Show that for any $f \in L^2(\mathbb{R}^n)$, there exists a unique solution $u \in H$ to $(\diamond)$.

d) Denote by Lf the unique solution $u \in H$ to $(\diamond)$ for $f \in L^2(\mathbb{R}^n)$. Show that the map $f \mapsto Lf$ is a compact, symmetric, linear operator $L : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$. Deduce that there exists an orthonormal basis $(w_k)_{k=1}^\infty$ for $L^2(\mathbb{R}^n)$ consisting of $w_k \in H$ satisfying:

$$(w_k, v)_H = \lambda_k (w_k, v)_{L^2} \text{ for all } v \in H, \qquad (\flat)$$

where $0 < \lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \cdots$, and $\lambda_k \to \infty$.

e) Show that if $w_k \in H$ satisfies ($\flat$), then in fact $w_k \in C^\infty(\mathbb{R}^n)$. Show further that $\hat{w}_k$ will also solve ($\flat$) with the same λ_k. Deduce that there exists an orthonormal basis for $L^2(\mathbb{R}^n)$, consisting of smooth functions, which diagonalises the Fourier–Plancherel transform.

f) (**) Show that $w \in H \cap C^\infty(\mathbb{R}^n)$ satisfies:

$$-\Delta w + |x|^2 w = \lambda w$$

for some $\lambda \in \mathbb{R}$ if and only if:

$$w(x) = H_{k_1}(x_1) \cdots H_{k_n}(x_n) e^{-\frac{1}{2}|x|^2},$$

where $x = (x_1, \ldots, x_n)$, $H_k(t)$ are the Hermite polynomials, and $\lambda = n + 2k_1 + \ldots 2k_n$. *[Hint: treat the case $n = 1$ first. You may wish to look up the simple harmonic oscillator in a textbook on quantum mechanics.]*

Exercise 3.13. Suppose that $u_0 \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and that $u(t, x)$ is the solution of the heat equation with initial data u_0. Explicitly, u is given by:

$$u(t, x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{u}_0(\xi) e^{-t|\xi|^2} e^{i\xi \cdot x} d\xi,$$

for $t > 0$.

a) Show that:

$$\|u(t, \cdot)\|_{L^2} \leqslant \|u_0\|_{L^2},$$

b) Show that:

$$u(t, x) = u_0 \star K_t(x)$$

where the *heat kernel* is given by:

$$K_t(x) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}.$$

c) Suppose that $u_0 \geqslant 0$. Show that $u \geqslant 0$, and:

$$\|u(t, \cdot)\|_{L^1} = \|u_0\|_{L^1}.$$

Exercise 3.14. Consider the free Schrödinger equation:

$$\begin{cases} u_t = i\Delta u & \text{in } (0, T) \times \mathbb{R}^n, \\ u = u_0 & \text{on } \{0\} \times \mathbb{R}^n \end{cases} \qquad (*)$$

Suppose $u_0 \in H^2(\mathbb{R}^n)$.

a) Show that $(*)$ admits a unique solution u such that

$$\boldsymbol{u} \in C^0([0,T);H^2(\mathbb{R}^n)) \cap C^1((0,T);L^2(\mathbb{R}^n)),$$

whose spatial Fourier-Plancherel transform is given by:

$$\hat{u}(t,\xi) = \hat{u}_0(\xi)e^{-it|\xi|^2}.$$

b) Show that:

$$\|u(t,\cdot)\|_{H^2(\mathbb{R}^n)} = \|u_0\|_{H^2(\mathbb{R}^n)}$$

*c) For $t > 0$, let $K_t \in L^1_{loc.}(\mathbb{R}^n)$ be given by:

$$K_t(x) = \frac{1}{(4\pi i t)^{\frac{n}{2}}} e^{\frac{i|x|^2}{4t}},$$

where for n odd we take the usual branch cut so that $i^{\frac{1}{2}} = e^{i\frac{\pi}{4}}$. For $\epsilon > 0$ set $K_t^\epsilon(x) = e^{-\epsilon|x|^2}K_t(x)$.

i) Show that $T_{K_t^\epsilon} \to T_{K_t}$ in $\mathscr{S}'$ as $\epsilon \to 0$.

ii) Show that if $\Re(\sigma) > 0$, then:

$$\int_{\mathbb{R}} e^{-\sigma x^2 - ix\xi}dx = \sqrt{\frac{\pi}{\sigma}}e^{-\frac{\xi^2}{4\sigma}}.$$

iii) Deduce that

$$\widehat{K_t^\epsilon}(\xi) = \left(\frac{1}{1+4it\epsilon}\right)^{\frac{n}{2}} e^{\frac{-it|\xi|^2}{1+4it\epsilon}}$$

iv) Conclude that:

$$\widehat{T_{K_t}} = T_{\tilde{K}_t},$$

where $\tilde{K}_t = e^{-it|\xi|^2}$.

*d) Suppose that $u \in \mathscr{S}(\mathbb{R}^n)$. Show that for $t > 0$:

$$u(t,x) = \int_{\mathbb{R}^n} u_0(y)K_t(x-y)dy,$$

and deduce that for $t > 0$:

$$\sup_{x\in\mathbb{R}^n} |u(t,x)| \leqslant \frac{1}{(4\pi t)^{\frac{n}{2}}} \|\hat{u}_0\|_{L^1}.$$

This type of estimate which shows us that (locally) solutions to the Schrödinger equation decay in time is known as a *dispersive estimate*.