Exercise 2.1. Let $\mathscr{P}$ be a separating family of seminorms on a vector space X. Show that a sequence $(x_k)_{k=1}^\infty$ with $x_k \in X$ converges to $x \in X$ in the topology $\tau_{\mathscr{P}}$ if and only if $p(x_k - x) \to 0$ for all $p \in \mathscr{P}$.

Exercise 2.2. Suppose that X is a Banach space, and let $(\Lambda_k)_{k=1}^\infty$ be a sequence with $\Lambda_k \in X'$. Show that:

$$\Lambda_k \to \Lambda \quad \Longrightarrow \quad \Lambda_k \rightharpoonup \Lambda \quad \Longrightarrow \quad \Lambda_k \overset{*}{\rightharpoonup} \Lambda.$$

(*) Show the stronger statement that $\tau_{w*} \subset \tau_w \subset \tau_s$, where $\tau_{w*}, \tau_w, \tau_s$ are the weak-$*$, weak and strong topologies on X' respectively.

Exercise 2.3. For a bounded measurable set $E \subset \mathbb{R}^n$ of positive measure, and any $f \in L^1_{loc.}(\mathbb{R}^n)$, define the mean of f on E to be:

$$⨍_E f(x)dx = \frac{1}{|E|} \int_E f(x)dx.$$

Suppose $1 < p < \infty$ and let $(f_j)_{j=1}^\infty$ be a bounded sequence of functions $f_j \in L^p(\mathbb{R}^n)$. Show that $f_j \rightharpoonup f$ for some $f \in L^p(\mathbb{R}^n)$ if and only if

$$⨍_E f_j(x)dx \to ⨍_E f(x)dx$$

for all bounded measurable sets $E \subset \mathbb{R}^n$ of positive measure.

Exercise 2.4. Suppose $(H, (\cdot, \cdot))$ is an infinite dimensional Hilbert space and let $(x_i)_{i=1}^\infty$ be a sequence with $x_i \in H$.

i) Show that $x_i \rightharpoonup x$ if and only if $(y, x_i) \to (y, x)$ for all $y \in H$.

ii) Show there exists a sequence such that $x_i \rightharpoonup 0$, but $x_i \not\to 0$.

iii) Suppose $x_i \rightharpoonup x$. Show that

$$||x|| \leqslant \liminf_{i\to\infty} ||x_i|| \,,$$

and $||x_i|| \to ||x||$ iff $x_i \to x$.

Exercise 2.5. Construct a bounded sequence $(f_i)_{i=1}^\infty$ of functions $f_i \in L^1(\mathbb{R})$ such that no subsequence is weakly convergent.

Exercise 2.6. Let X be a Banach space. Show, using the Hahn–Banach theorem, that X' separates points, i.e. for any $x, y \in X$, $x \neq y$ there exists $\Lambda \in X'$ with $\Lambda(x) \neq \Lambda(y)$.

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Exercise 2.7. let X be a reflexive Banach space, and suppose $Y \subset X$ is a closed subspace. Show that Y is reflexive.

Exercise 2.8. (*) Suppose X is a *separable* real Banach space. Prove the Hahn–Banach theorem on X *without* invoking the axiom of choice through Zorn's Lemma (or equivalent).

Exercise 2.9. a) Show that $\mathscr{S}$ is a vector subspace of $\mathscr{E}(\mathbb{R}^n)$. Show that if $\{\phi_j\}_{j=1}^{\infty}$ is a sequence of rapidly decreasing functions which tends to zero in $\mathscr{S}$, then $\phi_j \to 0$ in $\mathscr{E}(\mathbb{R}^n)$.

b) Show that $\mathscr{D}(\mathbb{R}^n)$ is a vector subspace of $\mathscr{S}$. Show that if $\{\phi_j\}_{j=1}^{\infty}$ is a sequence of compactly supported functions which tends to zero in $\mathscr{D}(\mathbb{R}^n)$ then $\phi_j \to 0$ in $\mathscr{S}$.

c) Give an example of a sequence $\{\phi_j\}_{j=1}^{\infty} \subset C_c^{\infty}(\mathbb{R}^n)$ such that

i) $\phi_j \to 0$ in $\mathscr{S}$, but ϕ_j has no limit in $\mathscr{D}(\mathbb{R}^n)$.

ii) $\phi_j \to 0$ in $\mathscr{E}(\mathbb{R}^n)$, but ϕ_j has no limit in $\mathscr{S}$.

Exercise 2.10. For each $X \in \{\mathscr{D}(\mathbb{R}^n), \mathscr{S}, \mathscr{E}(\mathbb{R}^n)\}$, suppose $\phi \in X$ and establish:

a) If $x_l \in \mathbb{R}^n$, $x_l \to 0$, then

$$\tau_{x_l}\phi \to \phi, \qquad \text{in } X \text{ as } l \to \infty,$$

where τ_x is the translation operator defined by $\tau_x\phi(y) := \phi(y - x)$.

b) If $h_l \in \mathbb{R}$, $h_l \to 0$, then

$$\Delta_i^{h_l}\phi \to D_i\phi, \qquad \text{in } X \text{ as } l \to \infty,$$

in X, where $\Delta_i^h\phi := h^{-1}\left[\tau_{-he_i}\phi - \phi\right]$ is the difference quotient.

Exercise 2.11. Suppose $u \in \mathscr{D}'(\mathbb{R})$ satisfies $Du = 0$. Show that u is a constant distribution, i.e. there exists $\lambda \in \mathbb{C}$ such that:

$$u[\phi] = \lambda \int_{\mathbb{R}} \phi(x)dx, \qquad \text{for all } \phi \in \mathscr{D}(\mathbb{R}).$$

(*) Extend the result to $\mathbb{R}^n$ for $n > 1$.
[Hint: Fix $\phi_0 \in \mathscr{D}(\mathbb{R})$ and show that any $\phi \in \mathscr{D}(\mathbb{R})$ may be written as $\phi(x) = \psi'(x) + c_\phi\phi_0(x)$ for some $\psi \in \mathscr{D}(\mathbb{R})$, $c_\phi \in \mathbb{C}$.]

Exercise 2.12. Let $X \in \{\mathscr{D}(\mathbb{R}^n), \mathscr{S}, \mathscr{E}(\mathbb{R}^n)\}$. For $u \in X'$, $x \in \mathbb{R}^n$, define $\tau_x u$ by $\tau_x u[\phi] = u[\tau_{-x}\phi]$ for all $\phi \in X$, and let $\Delta_i^h u = h^{-1}\left[\tau_{-he_i}u - u\right]$. Show that $\Delta_i^h u \to D_i u$ as $h \to 0$ in the weak-$*$ topology of X'.

Exercise 2.13. Suppose $u \in \mathscr{D}'(\mathbb{R}^n)$. Show that there exists a sequence of smooth functions $f_n \in C_c^{\infty}(\mathbb{R}^n)$ such that $T_{f_n} \to u$ in the weak-$*$ topology.