

**Exercise 1.1.** Suppose  $f, g : E \rightarrow \mathbb{C}$  are measurable functions on some measure space  $(E, \mathcal{E}, \mu)$ . Show that:

- a)  $\|fg\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}$  where  $1 \leq p, q, r \leq \infty$  satisfy  $p^{-1} + q^{-1} = r^{-1}$   
[You may wish to first establish the special case  $r = 1$ .]
- b)  $\|f + g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}$  for  $1 \leq p \leq \infty$ .

**Exercise 1.2.** a) Suppose that  $\mu(E) < \infty$ . Show that if  $f \in L^p(E, \mu)$ , then  $f \in L^q(E, \mu)$  for any  $1 \leq q \leq p$ , with

$$\|f\|_{L^q} \leq \mu(E)^{\frac{p-q}{qp}} \|f\|_{L^p}.$$

- b) Suppose that  $f \in L^{p_0}(E, \mu) \cap L^{p_1}(E, \mu)$  with  $p_0 < p_1 \leq \infty$ . For  $0 \leq \theta \leq 1$ , define  $p_\theta$  by

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

Show that  $f \in L^{p_\theta}(E, \mu)$  with

$$\|f\|_{L^{p_\theta}} \leq \|f\|_{L^{p_0}}^{1-\theta} \|f\|_{L^{p_1}}^\theta.$$

- c) Show that for  $p_1 \neq p_2$  we have  $L^{p_1}(\mathbb{R}^n) \not\subset L^{p_2}(\mathbb{R}^n)$ .

**Exercise 1.3.** Let  $\mathcal{R}_{\mathbb{Q}}$  be the set of rectangles of the form  $(a_1, b_1] \times \cdots \times (a_n, b_n]$  with  $a_i, b_i \in \mathbb{Q}$ , and let  $S_{\mathbb{Q}}$  be the set of functions of the form

$$s(x) = \sum_{k=1}^N (\alpha_k + i\beta_k) \mathbf{1}_{R_k}$$

for  $R_k \in \mathcal{R}_{\mathbb{Q}}$  and  $\alpha_k, \beta_k \in \mathbb{Q}$ . For  $1 \leq p < \infty$  show that  $S_{\mathbb{Q}}$  is dense in  $L^p(\mathbb{R}^n)$  and deduce that  $L^p(\mathbb{R}^n)$  is separable. Show that  $L^\infty(\mathbb{R}^n)$  is not separable.

[Hint: for the last part exhibit an uncountable subset  $X \subset L^\infty(\mathbb{R}^n)$  such that  $\|f - g\|_{L^\infty(\mathbb{R}^n)} \geq 1$  for any  $f, g \in X$ ,  $f \neq g$ ].

**Exercise 1.4.** a) Suppose  $1 \leq p \leq \infty$  and let  $q$  satisfy  $p^{-1} + q^{-1} = 1$ . Show that for a measurable function  $f : \mathbb{R}^n \rightarrow \mathbb{C}$ :

$$\|f\|_{L^p} = \sup \left\{ \left| \int_{\mathbb{R}^n} f(x)g(x)dx \right| : g \in L^q(\mathbb{R}^n), \|g\|_{L^q} \leq 1 \right\}.$$

- b) Now suppose  $p < \infty$  and assume  $F : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{C}$  is integrable. Set  $G(y) = \int_{\mathbb{R}^n} F(x, y) dx$ . Show that if  $\|g\|_{L^q} \leq 1$  then

$$\left| \int_{\mathbb{R}^n} G(y)g(y)dy \right| \leq \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} |F(x, y)|^p dy \right]^{\frac{1}{p}} dx.$$

Deduce Minkowski's integral inequality

$$\left[ \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} F(x, y) dx \right|^p dy \right]^{\frac{1}{p}} \leq \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} |F(x, y)|^p dy \right]^{\frac{1}{p}} dx.$$

**Exercise 1.5.** Suppose  $1 \leq p < \infty$ .

- a) Suppose  $f \in L^p(\mathbb{R}^n)$ . Show that

$$|\{x : |f(x)| > \lambda\}| \leq \frac{\|f\|_{L^p}^p}{\lambda^p}.$$

*This is known as Tchebychev's inequality, the  $p = 1$  case is Markov's inequality.*

- b) We say that a measurable  $f : \mathbb{R}^n \rightarrow \mathbb{C}$  is in *weak- $L^p(\mathbb{R}^n)$* , written  $f \in L^{p,w}(\mathbb{R}^n)$  if there exists a constant  $C$  such that

$$|\{x : |f(x)| > \lambda\}| \leq \frac{C^p}{\lambda^p}.$$

Show that  $L^p(\mathbb{R}^n) \subset L^{p,w}(\mathbb{R}^n)$ , and that the inclusion is proper.

**Exercise 1.6.** a) Let  $B_1, \dots, B_N$  be a finite collection of open balls in  $\mathbb{R}^n$ . Show that there exists a subcollection  $B_{i_1}, \dots, B_{i_k}$  of *disjoint* balls such that

$$\bigcup_{i=1}^N B_i \subset \bigcup_{j=1}^k (3B_{i_j}),$$

where  $3B$  is the ball with the same centre as  $B$  but three times the radius. Deduce

$$\left| \bigcup_{i=1}^N B_i \right| \leq 3^n \sum_{j=1}^k |B_{i_j}|.$$

- b) (\*) Suppose  $\{B_j : j \in J\}$  is an arbitrary collection of balls in  $\mathbb{R}^n$  such that each ball has radius at most  $R$ . Show that there exists a countable subcollection  $\{B_j : j \in J'\}$ ,  $J' \subset J$  of disjoint balls such that

$$\bigcup_{i \in J} B_i \subset \bigcup_{i \in J'} (5B_i).$$

*These are Wiener and Vitali's covering Lemmas, respectively.*

**Exercise 1.7.** Suppose  $\phi \in L^\infty(\mathbb{R}^n)$  satisfies  $\phi \geq 0$ ,  $\text{supp } \phi \subset B_1(0)$ , and  $\int_{\mathbb{R}^n} \phi \, dx = 1$ . Set  $\phi_\epsilon(x) = \epsilon^{-n} \phi(\epsilon^{-1}x)$ . Show that if  $f \in L^1(\mathbb{R}^n)$ , and  $x$  is a Lebesgue point of  $f$ ,

$$\phi_\epsilon \star f(x) \rightarrow f(x), \quad \text{as } \epsilon \rightarrow 0.$$

**Exercise 1.8.** Let  $S = \{\psi_{n,k}\}_{n,k \in \mathbb{Z}}$  be the Haar system, as defined in lectures.

a) Show that

$$\int_{\mathbb{R}} \psi_{n_1,k_1}(x) \psi_{n_2,k_2}(x) dx = \delta_{n_1 n_2} \delta_{k_1 k_2}.$$

b) Show that  $\mathbf{1}_I \in \overline{\text{Span } S}$  for any interval  $I$ , where the closure is understood with respect to the  $L^2$  norm.

c) Deduce that  $S$  is an orthonormal basis for  $L^2(\mathbb{R})$ .

**Exercise 1.9.** (\*) Suppose  $(E, \mathcal{E})$  is a measurable space, with finite measures  $\mu, \nu$ . Show that  $\nu$  may be uniquely written as  $\nu = \nu_a + \nu_s$ , for measures  $\nu_a, \nu_s$  such that  $\nu_s \perp \mu$  and  $\nu_a \ll \mu$ .

[Hint: Return to the proof of the Radon–Nikodym theorem, but drop the assumption that  $\nu \ll \mu$ ]

**Exercise 1.10.** Let  $X$  be a normed space. Show that  $X'$  equipped with its norm forms a Banach space. If  $\bar{X}$  is the completion of  $X$  with respect to the metric induced by its norm, show that  $X' = \bar{X}'$ .

**Exercise 1.11.** Let  $u : L^p(\mathbb{R}^n; \mathbb{R}) \rightarrow \mathbb{R}$  be a bounded, linear functional.

a) For  $f \in L^p(\mathbb{R}^n; \mathbb{R})$ ,  $f \geq 0$ , define

$$\tilde{u}(f) = \sup\{u(g) : g \in L^p(\mathbb{R}^n; \mathbb{R}), 0 \leq g \leq f\}.$$

Show that  $0 \leq \tilde{u}(f)$  and  $u(f) \leq \tilde{u}(f) \leq \|u\|_{L^{p'}} \|f\|_{L^p}$ , and establish

$$\tilde{u}(f + ag) = \tilde{u}(f) + a\tilde{u}(g)$$

for all  $f, g \in L^p(\mathbb{R}^n; \mathbb{R})$  with  $f, g \geq 0$  and  $a \in \mathbb{R}$ ,  $a > 0$ .

b) For  $f \in L^p(\mathbb{R}^n; \mathbb{R})$ , define  $w(f) = \tilde{u}(f^+) - \tilde{u}(f^-)$ , where  $f^+(x) = \max\{0, f(x)\}$ ,  $f^-(x) = \max\{0, -f(x)\}$ . Show that  $w$  is linear and bounded, and that  $w$  and  $w - u$  are positive.

c) Deduce that  $u = u_+ - u_-$ , where  $u_\pm$  are bounded, positive, linear functionals.

**Exercise 1.12.** Suppose  $X$  is a normed space, and  $V \subset X$  is a closed proper subspace of  $X$  and let  $0 < \alpha < 1$ . Show that there exists  $x \in X$  with  $\|x\| = 1$  such that  $\|x - y\| \geq \alpha$  for all  $y \in V$ . Deduce that the Bolzano–Weierstrass theorem does not hold if  $X$  is an infinite dimensional Banach space.

[The first result above is known as Riesz' Lemma]