

Exercise 1.1. Suppose $f, g : E \rightarrow \mathbb{C}$ are measurable functions on some measure space (E, \mathcal{E}, μ) . Show that:

- a) $\|fg\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}$ where $1 \leq p, q, r \leq \infty$ satisfy $p^{-1} + q^{-1} = r^{-1}$
[You may wish to first establish the special case $r = 1$.]
- b) $\|f + g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}$ for $1 \leq p \leq \infty$.

Exercise 1.2. a) Suppose that $\mu(E) < \infty$. Show that if $f \in L^p(E, \mu)$, then $f \in L^q(E, \mu)$ for any $1 \leq q \leq p$, with

$$\|f\|_{L^q} \leq \mu(E)^{\frac{p-q}{qp}} \|f\|_{L^p}.$$

b) Suppose that $f \in L^{p_0}(E, \mu) \cap L^{p_1}(E, \mu)$ with $p_0 < p_1 \leq \infty$. For $0 \leq \theta \leq 1$, define p_θ by

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

Show that $f \in L^{p_\theta}(E, \mu)$ with

$$\|f\|_{L^{p_\theta}} \leq \|f\|_{L^{p_0}}^{1-\theta} \|f\|_{L^{p_1}}^\theta.$$

c) Show that for $p_1 \neq p_2$ we have $L^{p_1}(\mathbb{R}^n) \not\subset L^{p_2}(\mathbb{R}^n)$.

Exercise 1.3. Let $\mathcal{R}_{\mathbb{Q}}$ be the set of rectangles of the form $(a_1, b_1] \times \cdots \times (a_n, b_n]$ with $a_i, b_i \in \mathbb{Q}$, and let $S_{\mathbb{Q}}$ be the set of functions of the form

$$s(x) = \sum_{k=1}^N (\alpha_k + i\beta_k) \mathbb{1}_{R_k}$$

for $R_k \in \mathcal{R}_{\mathbb{Q}}$ and $\alpha_k, \beta_k \in \mathbb{Q}$. For $1 \leq p < \infty$ show that $S_{\mathbb{Q}}$ is dense in $L^p(\mathbb{R}^n)$ and deduce that $L^p(\mathbb{R}^n)$ is separable. Show that $L^\infty(\mathbb{R}^n)$ is not separable.

[Hint: for the last part exhibit an uncountable subset $X \subset L^\infty(\mathbb{R}^n)$ such that $\|f - g\|_{L^\infty(\mathbb{R}^n)} \geq 1$ for any $f, g \in X$, $f \neq g$.]

Exercise 1.4. a) Suppose $1 \leq p \leq \infty$ and let q satisfy $p^{-1} + q^{-1} = 1$. Show that for a measurable function $f : \mathbb{R}^n \rightarrow \mathbb{C}$:

$$\|f\|_{L^p} = \sup \left\{ \left| \int_{\mathbb{R}^n} f(x)g(x)dx \right| : g \in L^q(\mathbb{R}^n), \|g\|_{L^q} \leq 1 \right\}.$$

b) Now suppose $p < \infty$ and assume $F : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{C}$ is integrable. Set $G(y) = \int_{\mathbb{R}^n} F(x, y) dx$. Show that if $\|g\|_{L^q} \leq 1$ then

$$\left| \int_{\mathbb{R}^n} G(y)g(y) dy \right| \leq \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} |F(x, y)|^p dy \right]^{\frac{1}{p}} dx.$$

Deduce Minkowski's integral inequality

$$\left[\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} F(x, y) dx \right|^p dy \right]^{\frac{1}{p}} \leq \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} |F(x, y)|^p dy \right]^{\frac{1}{p}} dx.$$

Exercise 1.5. Suppose $1 \leq p < \infty$.

a) Suppose $f \in L^p(\mathbb{R}^n)$. Show that

$$|\{x : |f(x)| > \lambda\}| \leq \frac{\|f\|_{L^p}^p}{\lambda^p}.$$

This is known as Tchebychev's inequality, the $p = 1$ case is Markov's inequality.

b) We say that a measurable $f : \mathbb{R}^n \rightarrow \mathbb{C}$ is in weak- $L^p(\mathbb{R}^n)$, written $f \in L^{p,w}(\mathbb{R}^n)$ if there exists a constant C such that

$$|\{x : |f(x)| > \lambda\}| \leq \frac{C^p}{\lambda^p}.$$

Show that $L^p(\mathbb{R}^n) \subset L^{p,w}(\mathbb{R}^n)$, and that the inclusion is proper.

Exercise 1.6. a) Let B_1, \dots, B_N be a finite collection of open balls in \mathbb{R}^n . Show that there exists a subcollection B_{i_1}, \dots, B_{i_k} of disjoint balls such that

$$\bigcup_{i=1}^N B_i \subset \bigcup_{j=1}^k (3B_{i_j}),$$

where $3B$ is the ball with the same centre as B but three times the radius. Deduce

$$\left| \bigcup_{i=1}^N B_i \right| \leq 3^n \sum_{j=1}^k |B_{i_j}|.$$

b) (*) Suppose $\{B_j : j \in J\}$ is an arbitrary collection of balls in \mathbb{R}^n such that each ball has radius at most R . Show that there exists a countable subcollection $\{B_j : j \in J'\}$, $J' \subset J$ of disjoint balls such that

$$\bigcup_{i \in J} B_i \subset \bigcup_{i \in J'} (5B_i).$$

These are Wiener and Vitali's covering Lemmas, respectively.

Exercise 1.7. Suppose $\phi \in L^\infty(\mathbb{R}^n)$ satisfies $\phi \geq 0$, $\text{supp } \phi \subset B_1(0)$, and $\int_{\mathbb{R}^n} \phi dx = 1$. Set $\phi_\epsilon(x) = \epsilon^{-n} \phi(\epsilon^{-1}x)$. Show that if $f \in L^1(\mathbb{R}^n)$, and x is a Lebesgue point of f ,

$$\phi_\epsilon \star f(x) \rightarrow f(x), \quad \text{as } \epsilon \rightarrow 0.$$

Exercise 1.8. Let $S = \{\psi_{n,k}\}_{n,k \in \mathbb{Z}}$ be the Haar system, as defined in lectures.

a) Show that

$$\int_{\mathbb{R}} \psi_{n_1, k_1}(x) \psi_{n_2, k_2}(x) dx = \delta_{n_1 n_2} \delta_{k_1 k_2}.$$

b) Show that $\mathbb{1}_I \in \overline{\text{Span } S}$ for any interval I , where the closure is understood with respect to the L^2 norm.

c) Deduce that S is an orthonormal basis for $L^2(\mathbb{R})$.

Exercise 1.9. (*) Suppose (E, \mathcal{E}) is a measurable space, with finite measures μ, ν . Show that ν may be uniquely written as $\nu = \nu_a + \nu_s$, for measures ν_a, ν_s such that $\nu_s \perp \mu$ and $\nu_a \ll \mu$.

[Hint: Return to the proof of the Radon–Nikodym theorem, but drop the assumption that $\nu \ll \mu$]

Exercise 1.10. Let X be a normed space. Show that X' equipped with its norm forms a Banach space. If \overline{X} is the completion of X with respect to the metric induced by its norm, show that $X' = \overline{X}'$.

Exercise 1.11. Let $u : L^p(\mathbb{R}^n; \mathbb{R}) \rightarrow \mathbb{R}$ be a bounded, linear functional.

a) For $f \in L^p(\mathbb{R}^n; \mathbb{R})$, $f \geq 0$, define

$$\tilde{u}(f) = \sup\{u(g) : g \in L^p(\mathbb{R}^n; \mathbb{R}), 0 \leq g \leq f\}.$$

Show that $0 \leq \tilde{u}(f)$ and $u(f) \leq \tilde{u}(f) \leq \|u\|_{L^{p'}} \|f\|_{L^p}$, and establish

$$\tilde{u}(f + ag) = \tilde{u}(f) + a\tilde{u}(g)$$

for all $f, g \in L^p(\mathbb{R}^n; \mathbb{R})$ with $f, g \geq 0$ and $a \in \mathbb{R}$, $a > 0$.

b) For $f \in L^p(\mathbb{R}^n; \mathbb{R})$, define $w(f) = \tilde{u}(f^+) - \tilde{u}(f^-)$, where $f^+(x) = \max\{0, f(x)\}$, $f^-(x) = \max\{0, -f(x)\}$. Show that w is linear and bounded, and that w and $w - u$ are positive.

c) Deduce that $u = u_+ - u_-$, where u_\pm are bounded, positive, linear functionals.

Exercise 1.12. Suppose X is a normed space, and $V \subset X$ is a closed proper subspace of X and let $0 < \alpha < 1$. Show that there exists $x \in X$ with $\|x\| = 1$ such that $\|x - y\| \geq \alpha$ for all $y \in V$. Deduce that the Bolzano–Weierstrass theorem does not hold if X is an infinite dimensional Banach space.

[The first result above is known as Riesz' Lemma]