II Algebraic Topology // Example Sheet 4

1. Show that if $n \neq m$ then $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic.

2. For each of the following exact sequences of abelian groups and homomorphisms say as much as possible about the unknown group A and homomorphism α.

   (i) $0 \longrightarrow \mathbb{Z}/2 \longrightarrow A \longrightarrow \mathbb{Z} \longrightarrow 0$,

   (ii) $0 \longrightarrow \mathbb{Z}/2 \longrightarrow A \longrightarrow \mathbb{Z}/2 \longrightarrow 0$,

   (iii) $0 \longrightarrow A \longrightarrow \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \longrightarrow \mathbb{Z}/2 \longrightarrow 0$,

   (iv) $0 \longrightarrow \mathbb{Z}/3 \longrightarrow A \longrightarrow \mathbb{Z}/2 \longrightarrow \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \longrightarrow 0$.

3. Consider a commutative diagram

$$\begin{array}{ccccccccc}
A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & A_4 & \xrightarrow{f_4} & A_5 \\
\downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_2} & & \downarrow{\scriptstyle h_3} & & \downarrow{\scriptstyle h_4} & & \downarrow{\scriptstyle h_5} \\
B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & B_3 & \xrightarrow{g_3} & B_4 & \xrightarrow{g_4} & B_5
\end{array}$$

   in which the rows are exact and each square commutes. If h_1, h_2, h_4, and h_5 are isomorphisms, show that h_3 is too.

4. Let K be a simplicial complex in $\mathbb{R}^m$, and consider this as lying inside $\mathbb{R}^{m+1}$ as the vectors of the form $(x_1, \ldots, x_n, 0)$. Let $e_+ = (0, \ldots, 0, 1) \in \mathbb{R}^{m+1}$ and $e_- = (0, \ldots, 0, -1) \in \mathbb{R}^{m+1}$. The *suspension* of K is the simplicial complex in $\mathbb{R}^{m+1}$

$$SK := K \cup \{\langle v_0, \ldots, v_n, e_+\rangle, \langle v_0, \ldots, v_n, e_-\rangle \mid \langle v_0, \ldots, v_n\rangle \in K\}.$$

   (i) Show that SK is a simplicial complex, and that if $|K| \cong S^n$ then $|SK| \cong S^{n+1}$.

   (ii) Using the Mayer–Vietoris sequence, show that if K is connected then $H_0(SK) \cong \mathbb{Z}$, $H_1(SK) = 0$, and $H_i(SK) \cong H_{i-1}(K)$ if $i \geq 2$.

   (iii) If $f : K \to K$ is a simplicial map, let $Sf : SK \to SK$ be the unique simplicial map which agrees with f on the subcomplex K and fixes the points e_+ and e_-. Show that under the isomorphism in (ii), the maps f_* and Sf_* agree. *[Hint: It may help to describe the isomorphism in (ii) at the level of chains.]*

   (iv) Deduce that for every $n \geq 1$ and $d \in \mathbb{Z}$ there is a map $f : S^n \to S^n$ so that f_* induces multiplication by d on $H_n(S^n) \cong \mathbb{Z}$.

5. If K is a simplicial complex with $H_i(K) \cong \mathbb{Z}^r \oplus F$, for F a finite abelian group, show that $H_i(K; \mathbb{Q}) \cong \mathbb{Q}^r$. *[Hint: There is a chain map $C_\bullet(K) \to C_\bullet(K; \mathbb{Q})$.]*

6. By describing a triangulation of S^n which is preserved under the antipodal map, show that $\mathbb{RP}^n$ has a triangulation. *[Be careful that the triangulation you describe actually comes from a simplicial complex! Some subdivision may be necessary.]* Using the Mayer–Vietoris sequence, show that there is an exact sequence

$$0 \longrightarrow H_n(\mathbb{RP}^n) \longrightarrow \mathbb{Z} \longrightarrow H_{n-1}(\mathbb{RP}^{n-1}) \longrightarrow H_{n-1}(\mathbb{RP}^n) \longrightarrow 0$$

and that $H_i(\mathbb{RP}^{n-1}) \to H_i(\mathbb{RP}^n)$ is an isomorphism for $i < n-1$. Hence show that

$$H_i(\mathbb{RP}^n) \cong \begin{cases} \mathbb{Z} & \text{if } i = 0 \text{ or if } i = n \text{ and } n \text{ is odd} \\ \mathbb{Z}/2 & \text{if } i \text{ is odd and } 0 < i < n \\ 0 & \text{otherwise.} \end{cases}$$

Deduce that $\mathbb{RP}^{2k}$ does not retract onto $\mathbb{RP}^{2k-1}$, and that any map $f : \mathbb{RP}^{2k} \to \mathbb{RP}^{2k}$ has a fixed point.

7. Let A be a 2×2 matrix with entries in $\mathbb{Z}$. Show that the linear map $A : \mathbb{R}^2 \to \mathbb{R}^2$ peserves the equivalence relation $(a, b) \sim (a', b') \iff (a - a', b - b') \in \mathbb{Z}^2$, and so induces a continuous map f_A from the torus T to itself. Compute the effect of the continuous map f_A on the homology of T.

8. For triangulated surfaces X and Y, let $X \# Y$ be the surface obtained by cutting out a 2-simplex from both X and Y and then gluing together the two copies of $\partial \Delta^2$ formed. Use the Mayer–Vietoris sequence to compute the homology of $\Sigma_g \# S_n$, and hence deduce that it is homeomorphic to S_{n+2g}.

9. Let $p : \widetilde{X} \to X$ be a finite-sheeted covering space, and $h : |K| \to X$ a triangulation. Show that there is an $r \geq 1$ and triangulation $g : |L| \to \widetilde{X}$ so that the composition $h^{-1} \circ p \circ g : |L| \to |K^{(r)}|$ is a simplicial map. If p has n sheets, show that $\chi(\widetilde{X}) = n \cdot \chi(X)$. Hence show that Σ_g is a covering space of Σ_h if and only if $\frac{1-g}{1-h}$ is an integer. *[Hint: If $g = 1 + k \cdot (h-1)$, show that $\mathbb{Z}/k$ acts freely and properly discontinuously on a particular orientable surface of genus g, and identify the quotient.]*

10. Let $p : S^{2k} \to X$ be a covering map, $G = \pi_1(X, [x_0])$, and recall that G then acts freely on S^{2k}. Show that for any $g \in G$ the map $g_* : H_{2k}(S^{2k}) \to H_{2k}(S^{2k})$ is multiplication by -1. Deduce that G is either trivial or $\mathbb{Z}/2$, and that $\mathbb{RP}^{2k}$ is not a proper covering space of any other space.

11. If $f : K \to K$ is a simplicial isomorphism, let $X \subset |K|$ be the fixed set of $|f|$ i.e. $\{x \in |K| \mid |f|(x) = x\}$. Show that the Lefschetz number $L(f)$ is equal to $\chi(X)$. *[Hint: Barycentrically subdivide K so that X is the polyhedron of a sub simplicial complex.]*

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