Algebraic Topology **Michaelmas 2021**

Example Sheet 4

1. For each of the following exact sequences of abelian groups and homomorphisms say as much as possible about the unknown group G and homomorphism α.

   (a) $0 \longrightarrow \mathbb{Z}/2 \longrightarrow G \longrightarrow \mathbb{Z} \longrightarrow 0$,

   (b) $0 \longrightarrow \mathbb{Z} \longrightarrow G \longrightarrow \mathbb{Z}/2 \longrightarrow 0$,

   (c) $0 \longrightarrow G \longrightarrow \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \longrightarrow \mathbb{Z}/2 \longrightarrow 0$,

   (d) $0 \longrightarrow \mathbb{Z}/3 \longrightarrow G \longrightarrow \mathbb{Z}/2 \longrightarrow \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \longrightarrow 0$.

2. Consider a commutative diagram

$$\begin{tikzcd}
A_5 \arrow[r, "f_5"] \arrow[d, "h_5"] & A_4 \arrow[r, "f_4"] \arrow[d, "h_4"] & A_3 \arrow[r, "f_3"] \arrow[d, "h_3"] & A_2 \arrow[r, "f_2"] \arrow[d, "h_2"] & A_1 \arrow[d, "h_1"] \\
B_5 \arrow[r, "g_5"] & B_4 \arrow[r, "g_4"] & B_3 \arrow[r, "g_3"] & B_2 \arrow[r, "g_2"] & B_1
\end{tikzcd}$$

   in which the rows are exact and each square commutes.

   (a) Using the maps f_i, g_i, and h_i, construct a chain complex (C_*, d) with $C_i = A_i \oplus B_{i+1}$. By considering the subcomplex generated by the B_i, show that $H_3(C) = H_2(C) = 0$.

   (b) Now suppose that h_1, h_2, h_4, and h_5 are isomorphisms. By considering subcomplexes defined by vertical lines, show that $H_i(C) = 0$ for $i \neq 2, 3$. Deduce that $H_i(C) = 0$ for all i and that h_3 is an isomorphism.

3. Use the Mayer-Vietoris sequence to compute $H_*(Y)$ when

   (a) Y is obtained by deleting the interiors of r distinct n-simplices from a triangulation of S^n.

   (b) $Y = X/\sim$, where X is the disjoint union of two copies of T^2, and $\sim$ is the minimal equivalence relation for which $(1, \theta)$ in the first copy of T^2 is identified with $(\theta, 1)$ in the second copy of T^2.

   (c) $Y = A \cup B$, where

$$A = \{(x_1, x_2, x_3, x_4, 0, 0) \in \mathbb{R}^6 \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}$$
$$B = \{(x_1, x_2, 0, 0, x_3, x_4) \in \mathbb{R}^6 \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}.$$

   You may assume that A and B can be triangulated so that $A \cap B$ is a subcomplex of both A and B.

4. Let $p : \widehat{X} \to X$ be a finite-sheeted covering space, and $h : |K| \to X$ a triangulation. Show that there is an $r \geq 1$ and triangulation $g : |L| \to \widehat{X}$ so that the composition $h^{-1} \circ p \circ g : |L| \to |K^{(r)}|$ is a simplicial map. If p has n sheets, show that $\chi(\widehat{X}) = n \cdot \chi(X)$. Hence show that Σ_g is a covering space of Σ_h if and only if $\frac{1-g}{1-h}$ is an integer.

5. Let $\alpha : S^n \to S^n$ be the antipodal map. Compute the Lefschetz number $L(\alpha)$. When is α homotopic to the identity?

6. Let $p : S^{2k} \to X$ be a covering map, $G = \pi_1(X, x_0)$, and recall that G acts freely on S^{2k} via deck transformations. Show that, for any $g \in G$, the map $g_* : H_{2k}(S^{2k}) \to H_{2k}(S^{2k})$ is multiplication by -1. Deduce that G is either trivial or $\mathbb{Z}/2$, and that $\mathbb{RP}^{2k}$ is not a proper covering space of any other space.

7. Let $f : K \to K$ be a simplicial isomorphism, and let $X \subset |K|$ be the fixed-point set of $|f|$ (i.e. $\{x \in |K| \mid |f|(x) = x\}$). Show that the Lefschetz number $L(f)$ is equal to $\chi(X)$.

   [Hint: Show that $X = |L|$, where L is subcomplex of $K^{(r)}$.]

8. Suppose K is a simplicial complex and that v is a vertex of K. Fix a generator x of $H_1(S^1)$. Show that the map $\Omega_1(|K|, v) \to H_1(|K|)$ given by $\gamma \mapsto \overline{\gamma}_*(x)$ descends to a homomorphism $H : \pi_1(|K|, v) \to H_1(|K|)$. * If $|K|$ is path connected, show that H is surjective. Deduce that the abelianization $\pi_1(|K|, v)^{\mathrm{ab}}$ surjects onto $H_1(|K|)$.

9. Let A be a 2×2 matrix with entries in $\mathbb{Z}$. Show that the linear map $A : \mathbb{R}^2 \to \mathbb{R}^2$ preserves the equivalence relation $(a, b) \sim (a', b') \iff (a - a', b - b') \in \mathbb{Z}^2$, and so induces a continuous map $f_A : T^2 \to T^2$. Compute $(f_A)_* : H_1(T^2) \to H_1(T^2)$. [You may find it helpful to use problem 8.]

10. * By considering the action of A_5 on the icosahedron, construct a space P with a degree 60 covering map $p : SO(3) \to P$. Show that P is a compact 3-manifold; *i.e.* it is compact, Hausdorff, and every point has an open neighborhood homeomorphic to $\mathbb{R}^3$. Using ES 1 Q 14, show that $G = \pi_1(P)$ is a group of order 120. Is $G \cong S_5$? Show that $G^{ab} = 1$, and deduce using Q 8 that $H_1(P) = 0$.

    Taking as given that P admits a triangulation of dimension 3 in which every 2-dimensional simplex is the face of exactly two 3-dimensional simplices, show that $H_3(P) = \mathbb{Z}$ and $b_2(P) = 0$, so $H_*(P; \mathbb{Q}) \cong H_*(S^3; \mathbb{Q})$. (In fact if M is a compact orientable n-manifold, $H_{n-1}(M)$ is torsion free, so $H_*(P) \cong H_*(S^3)$).)

 jar60@cam.ac.uk