Algebraic Topology **Michaelmas 2021**

Example Sheet 2

1. What is the universal cover of $S^1 \vee S^2$? Draw a picture.

2. Let $S = (S^1 \times 1) \cup (1 \times S^1) \subset S^1 \times S^1 = T^2$. Show that T^2 does not retract onto S.

3. If (X, x_0) is a pointed space, prove that $(X, x_0) \vee (I, 0)$ is homotopy equivalent to X. Deduce that any finite tree is contractible, and that the regular 4-valent tree is simply connected. *Show further that it is contractible.

4. Draw pictures of all the connected degree 2 covering spaces of $S^1 \vee S^1$. Show that they are all normal coverings, and describe the action of the group of deck transformations in each case. Now do the same thing for the connected degree 3 covering spaces of $S^1 \vee S^1$. Which of them are normal coverings?

5. Consider $X = S^1 \vee S^1$ with basepoint x_0 the wedge point, which has $\pi_1(X, x_0) = \langle a, b \rangle$ where a and b are given by the usual two loops. Describe covering spaces associated to:

   (a) $\langle\langle a \rangle\rangle$, the normal subgroup generated by a;

   (b) $\langle a \rangle$, the subgroup generated by a;

   (c) the kernel of the homomorphism $\phi : \langle a, b \rangle \to \mathbb{Z}/4\mathbb{Z}$ given by $\phi(a) = 1$ and $\phi(b) = -1$.

   What are the fundamental groups of these covering spaces?

6. Show that for any $n > 0$, the free group F_2 contains a subgroup isomorphic to F_n. Show that if $m < n$, there is no surjective homomorphism $\phi : F_m \to F_n$.

7. Let X be a Hausdorff space, and G a group acting on X by homeomorphisms, *freely* (i.e. if $g \in G$ satisfies $g \cdot x = x$ for some $x \in X$, then $g = 1$) and *properly discontinuously* (i.e. each $x \in X$ has an open neighbourhood $U \ni x$ such that $\{g \in G \mid g(U) \cap U \neq \varnothing\}$ is finite).

   (a) Show that the quotient map $X \to G \backslash X$ is a covering map.

   (b) Deduce that if X is simply-connected then for any point $[x] \in G \backslash X$ we have an isomorphism of groups $\pi_1(G \backslash X, [x]) \cong G$.

   (c) Hence show that for any $m \geq 2$ there is a space X with fundamental group $\mathbb{Z}/m$ and universal cover S^3. [*Hint: Consider S^3 as the unit sphere in $\mathbb{C}^2$.*]

8. Show that the groups

$$G = \langle a, b \mid a^3 b^{-2} \rangle \quad \text{and} \quad H = \langle x, y \mid xyxy^{-1}x^{-1}y^{-1} \rangle$$

are isomorphic. By constructing surjective homomorphisms to appropriate groups, show that this group is infinite and non-abelian.

9. Show that the Klein bottle is homeomorphic to a space obtained by attaching a 2-cell to $S^1 \vee S^1$. Use this to give a presentation of its fundamental group with two generators and one relator. Give an explicit isomorphism between the group defined by your presentation and the group in Q13 of Sheet 1.

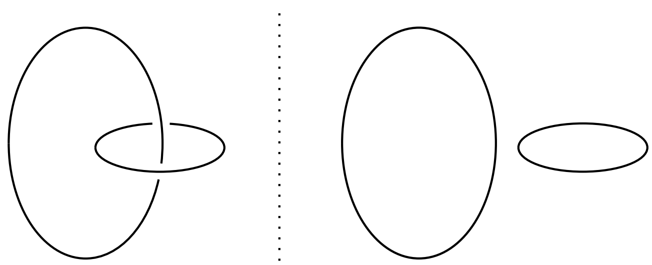
10. If $Y = \mathbb{RP}^2 \vee \mathbb{RP}^2$ and y_0 is the wedge point, let $G = \pi_1(Y, y_0)$, so

$$G \simeq \mathbb{Z}/2 * \mathbb{Z}/2 \simeq \langle a, b \mid a^2, b^2 \rangle.$$

(a) Describe the covering space of Y corresponding to $\ker \phi$, where $\phi : G \to \mathbb{Z}/2$ is given by $\phi(a) = 1$ and $\phi(b) = 0$. Hence show that $\ker \phi \simeq G$.

(b) Draw a picture of the universal cover $\widetilde{Y}$, and describe the actions of the elements of the deck group corresponding to a, b, and ab on it. Deduce that ab has infinite order in G.

11. Consider the following configurations of pairs of circles in S^3 (we have drawn them in $\mathbb{R}^3$; add a point at infinity).



By computing the fundamental groups of the complements of the circles, show there is no homeomorphism of S^3 taking one configuration to the other.

12.* View S^3 as the set $\{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\}$, and let $K \subset S^3$ be the set

$$K = \{(z, w) \in S^3 \mid z^2 = w^3\}.$$

(a) Show that K is homeomorphic to S^1.

(b) Use the Seifert-Van Kampen theorem to show that $\pi_1(S^3 - K, x_0)$ is isomorphic to the group G in question 7. [*Hint: Let* $T = \{(z, w) \in S^3 \mid |z|^2 = |w|^3\}$. *Split* $S^3 - K$ *into two pieces along* $T - K$.]

(c) Let $U = \{(z, 0) \mid |z| = 1\}$ be the *unknot* in S^3. Show that there is no homeomorphism $f : S^3 \to S^3$ with $f(K) = U$.

(d) We can identify $S^3 - (0, 0, 0, 1)$ with $\mathbb{R}^3$. Using this identification, sketch the images of U and K in $\mathbb{R}^3$. [*Hint: first sketch* T].