

# Algebraic Topology, Examples 1

Michaelmas 2020

1. Let  $a : S^n \rightarrow S^n$  be the antipodal map,  $a(x) = -x$ . Show that  $a$  is homotopic to the identity map when  $n$  is odd. [Try  $n = 1$  first.]
2. Let  $f : S^1 \rightarrow S^1$  be a map which is not homotopic to the identity map. Show that there exists an  $x \in S^1$  such that  $f(x) = x$ , and a  $y \in S^1$  so that  $f(y) = -y$ .
3. Suppose that  $f : X \rightarrow Y$  is a map for which there exist maps  $g, h : Y \rightarrow X$  such that  $g \circ f \simeq \text{Id}_X$  and  $f \circ h \simeq \text{Id}_Y$ . Show that  $f$ ,  $g$ , and  $h$  are homotopy equivalences.
4. Show that a retract of a contractible space is contractible.
5. Show that if a space  $X$  deformation retracts to a point  $x_0 \in X$ , then for every open neighbourhood  $x_0 \in U$  there exists a smaller open neighbourhood  $x_0 \in V \subset U$  such that the inclusion  $(V, x_0) \hookrightarrow (U, x_0)$  is homotopic *relative to*  $\{x_0\}$  to the constant map.
6. Construct a space which contains both the annulus  $S^1 \times I$  and the Möbius band as deformation retracts.
7. For  $m < n$ , consider  $S^m$  as a subspace of  $S^n$  given by

$$\{(x_1, x_2, \dots, x_{m+1}, 0, \dots, 0) \mid \sum x_i^2 = 1\}.$$

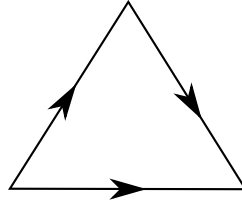
Show that the complement  $S^n - S^m$  is homotopy equivalent to  $S^{n-m-1}$ .

8. A space is called *locally path connected* if for every point  $x \in X$  and every neighbourhood  $U \ni x$ , there exists a smaller neighbourhood  $V$  of  $x$  (i.e.  $x \in V \subset U$ ) which is path connected. Show that a locally path connected space which is connected is also path connected.
9. For a map  $f : S^{n-1} \rightarrow X$  we define the *space obtained by attaching an  $n$ -cell to  $X$  along  $f$*  to be the quotient space

$$X \cup_f D^n := (X \amalg D^n) / \sim$$

where  $\sim$  is the smallest equivalence relation containing  $b \sim f(b)$  for every  $b \in S^{n-1} \subset D^n$ . Show that if  $f, f' : S^{n-1} \rightarrow X$  are homotopic maps then  $X \cup_f D^n \simeq X \cup_{f'} D^n$ .

**10.** The *dunce cap* is the space obtained from a solid triangle by gluing the edges together as shown.



Show that this space is contractible. [Hint: use the previous question.]

**11.** Show that the Möbius band does not retract onto its boundary.

**12.** For based spaces  $(X, x_0)$  and  $(Y, y_0)$  show there is an isomorphism

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

**13.** Construct a covering map  $\pi : \mathbb{R}^2 \rightarrow K$  of the Klein bottle, and hence show that  $\pi_1(K, k_0)$  is isomorphic to the group  $G$  with elements  $(m, n) \in \mathbb{Z}^2$  and group operation

$$(m, n) * (p, q) = (m + (-1)^n \cdot p, n + q).$$

Show that  $K$  has a covering space homeomorphic to the torus  $S^1 \times S^1$ , but that the torus does not have a covering space homeomorphic to  $K$ .

**14.\*** A *topological group* consists of a set  $G$  equipped with both a topology and a group structure, so that the inversion map  $i : G \rightarrow G$  (that sends  $g \mapsto g^{-1}$ ) and the multiplication map  $m : G \times G \rightarrow G$  (that sends  $(g, h) \mapsto gh$ ) are continuous. (Here,  $G \times G$  is equipped with the product topology.)

Let  $G$  be a path-connected, locally-path-connected topological group, and  $p : \widehat{G} \rightarrow G$  be a path-connected covering space. Let  $e$  be the identity of  $G$  and  $\epsilon \in p^{-1}(e)$ .

- (i) Show that  $\widehat{G}$  has a unique structure of a topological group with unit  $\epsilon$  so that  $p$  is a continuous homomorphism.
- (ii) Show that  $\text{Ker}(p) \subset \widehat{G}$  lies in the centre of  $\widehat{G}$ .
- (iii) Show that  $SO(3)$ , the group of rotations of  $\mathbb{R}^3$  (or equivalently of orthogonal  $3 \times 3$  matrices of determinant 1), is homeomorphic to the projective space  $\mathbb{RP}^3$ .
- (iv) Together, (i) and (iii) give a universal cover  $\widetilde{SO(3)}$  homeomorphic to  $S^3$ . Identify this group with a well-known matrix group.