The proof of the Homotopy Lifting Lemma.

Lemma -1.1 (Homotopy-lifting lemma)**.** *Let $p : \widehat{X} \to X$ be a covering map and let $f_0 : Y \to X$ be a map from a locally path-connected space Y. Let $F : Y \times I \to X$ be a homotopy with $F(y,0) = f_0(y)$ for all y, and let $\hat{f}_0 : Y \to \widehat{X}$ be a lift of f_0. There is a unique lift $\widehat{F}$ of F to $\widehat{X}$ so that $\widehat{F}(\cdot, 0) = \hat{f}_0$.*

Proof. For each $y \in Y$, the homotopy F defines a path

$$\gamma_y(t) = F(y,t)$$

from $f_0(y)$. By the path-lifting lemma, each γ_y lifts at $\hat{f}_0(y)$ to a path $\hat{\gamma}_y$ in $\widehat{X}$. By the uniqueness of lifts, we must have

$$\widehat{F}(y,t) = \hat{\gamma}_y(t)$$

for all $y \in Y$ and $t \in I$. It remains to prove that $\widehat{F}$ is continuous. To do this, we define a different lift $\widetilde{F}$ of F which is continuous by definition, and prove that the two lifts agree.

Consider $y_0 \in Y$. For any t, $F(y_0, t)$ has an evenly covered neighbourhood U_t in X. By compactness of $\{y_0\} \times I$, we may take finitely many intervals $\{J_i\}$ that cover I and a path-connected neighbourhood V of y_0 so that, for each i, $F(V \times J_i)$ is contained in some evenly covered set U_i. Let U_{δ_i} be the unique slice of $p^{-1}(U_i)$ such that $\widehat{F}(\{y_0\} \times J_i) \subseteq U_{\delta_i}$.

For any $(y,t) \in V \times I$, we now define

$$\widetilde{F}(y,t) = p_{\delta_i}^{-1} \circ F(y,t)$$

whenever $t \in J_i$. We need to check that this is well defined. Suppose, therefore, that $t \in J_i \cap J_j$. Let α be a path in V from y_0 to y and let

$$\alpha_t(s) = F(\alpha(s), t)\ .$$

Then $p_{\delta_i}^{-1} \circ \alpha_t$ is a lift of α_t at $p_{\delta_i}^{-1} \circ \alpha_t(0)$ and, likewise, $p_{\delta_j}^{-1} \circ \alpha_t$ is a lift of α_t at $p_{\delta_j}^{-1} \circ \alpha_t(0)$. But

$$p_{\delta_i}^{-1} \circ \alpha_t(0) = \widehat{F}(y_0, t) = p_{\delta_j}^{-1} \circ \alpha_t(0)$$

so, by uniqueness of lifts, $p_{\delta_i}^{-1} \circ \alpha_t(1) = p_{\delta_j}^{-1} \circ \alpha_t(1)$. Therefore, $p_{\delta_i}^{-1} \circ F(y,t) = p_{\delta_j}^{-1} \circ F(y,t)$, which proves that $\widetilde{F}$ is well defined.

Since V is connected and $\widetilde{F}(\cdot, 0)$ is a lift of f_0 that agrees with $\hat{f}_0$ at y_0, we have that $\widetilde{F}(y, 0) = \hat{f}_0(y)$ for all $y \in V$, by uniqueness of lifts. Now, for each $y \in V$, $\widetilde{F}(y, \cdot)$ is a lift of γ_y at $\hat{f}_0(y)$, and so $\widetilde{F}(y, t) = \hat{\gamma}_y(t)$ by uniqueness of lifts. Therefore, $\widetilde{F}$ and $\widehat{F}$ agree on $V \times I$. But $\widetilde{F}$ is continuous by construction, so $\widehat{F}$ is too. □