Algebraic Topology, Examples 2

Michaelmas 2018

The *wedge* of two spaces $X \vee Y$ is the quotient space obtained from the disjoint union $X \sqcup Y$ by identifying two points $x \in X$ and $y \in Y$.

Questions marked with a (*) are optional.

1. A *graph* X is defined as follows. Consider two discrete spaces V and E with continuous maps $\iota, \tau : E \to V$. Then

$$X = (V \sqcup (E \times I))/\sim$$

where $\sim$ is the smallest equivalence relation so that $(e, 0) \sim \iota(e)$ and $(e, 1) \sim \tau(e)$.

   (a) Show that S^1 is homeomorphic to a graph.

   (b) Show that $S^1 \vee S^1$ is homeomorphic to a graph.

   (c) Draw all the covering spaces of S^1 of degree 2 or 3.

   (d) Draw all the covering spaces of $S^1 \vee S^1$ of degree 2 or 3.

2. Prove that every covering space of a graph is a graph.

3. Let X be a Hausdorff space, and G a group acting on X by homeomorphisms, *freely* (i.e. if $g \in G$ satisfies $g \cdot x = x$ for some $x \in X$, then $g = 1$) and *properly discontinuously* (i.e. each $x \in X$ has an open neighbourhood $U \ni x$ such that $\{g \in G \mid g(U) \cap U \neq \varnothing\}$ is finite).

   (a) Show that the quotient map $X \to G\backslash X$ is a covering map.

   (b) Deduce that if X is simply-connected then for any point $[x] \in G\backslash X$ we have an isomorphism of groups $\pi_1(G\backslash X, [x]) \cong G$.

   (c) Hence show that for any $m \geq 2$ there is a space X with fundamental group $\mathbb{Z}/m$ and universal cover S^3. *[Hint: Consider S^3 as the unit sphere in $\mathbb{C}^2$. You may use without proof the fact that S^3 is simply connected.*

4. Consider $X = S^1 \vee S^1$ with basepoint x_0 the wedge point, which has $\pi_1(X, x_0) = \langle a, b \rangle$ where a and b are given by the usual two loops. Describe covering spaces associated to:

   (a) $\langle\langle a \rangle\rangle$, the normal subgroup generated by a;

   (b) $\langle a \rangle$, the subgroup generated by a;

(c) the kernel of the homomorphism $\phi : \langle a, b \rangle \to \mathbb{Z}/4\mathbb{Z}$ given by $\phi(a) = [1]$ and $\phi(b) = [3] = [-1]$.

Show that the free group on two letters contains a copy of itself as a proper subgroup.

5. Show that the groups

$$G = \langle a, b \,|\, a^3 b^{-2} \rangle \quad \text{and} \quad H = \langle x, y \,|\, xyxy^{-1}x^{-1}y^{-1} \rangle$$

are isomorphic. Show that this group is non-abelian and infinite.

[Hint: Construct surjective homomorphisms to appropriate groups.]

6. What is the universal cover of $S^1 \vee S^2$?

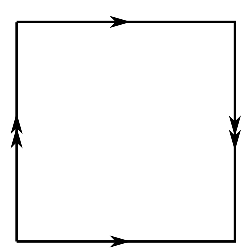
7. Let $X = S^1 \vee S^1$ as in question 4. Consider the space Y obtained from X by attaching 2-cells along loops in the homotopy classes a^2 and b^2, so that

$$\pi_1(Y, x_0) \cong \langle a, b \,|\, a^2, b^2 \rangle.$$

(a) Construct (in pictures) the covering space of Y corresponding to the subgroup $\langle a \,|\, a^2 \rangle$.

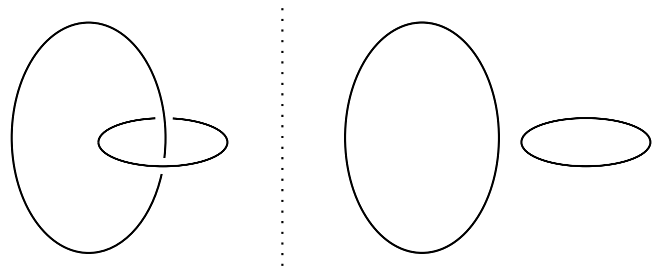
(b) Construct (in pictures) the covering space of Y corresponding to the kernel of the homomorphism $\phi : \langle a, b \,|\, a^2, b^2 \rangle \to \mathbb{Z}/2$ given by $\phi(a) = 1$ and $\phi(b) = 0$. Hence show that $\mathrm{Ker}(\phi)$ is isomorphic to $\langle a, b \,|\, a^2, b^2 \rangle$.

8. The *Klein bottle* is the surface obtained from the following identification diagram.



Explain how to construct the Klein bottle by attaching a 2-cell to a graph. Deduce that its fundamental group has a presentation $\langle a, b \,|\, baba^{-1} \rangle$, and show this is isomorphic to the group in Q13 of Sheet 1.

9. Consider the following configurations of pairs of circles in S^3 (we have drawn them in $\mathbb{R}^3$; add a point at infinity).



By computing the fundamental groups of the complements of the circles, show there is no homeomorphism of S^3 taking one configuration to the other.

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10. (*) In question (1d) you drew various covering maps $p: \widehat{Y} \to S^1 \vee S^1$. Which of these arose from the construction in Question 3? That is, for which $\widehat{Y}$ is there a group G acting freely and properly discontinuously so that p is the quotient map $\widehat{Y} \to G\backslash\widehat{Y}$?

11. (*) A *tree* is a simply connected graph. A *star* is a tree with a vertex x_0 such that one end of each edge is attached to x_0. A *leaf* of a tree is a vertex attached to only one edge. Prove that every tree is homotopy equivalent to a star, relative to its leaves.

12. (*) A tree T which is a subgraph of a graph X is *maximal* if it contains every vertex. You may assume that every graph has a maximal tree.

    (a) If $T \subset G$ is a tree, show that the quotient map $G \to G/T$ is a homotopy equivalence. Hence show that every connected graph is homotopy equivalent to a graph with a single vertex. *[Hint: Use question 11.]*

    (b) Show that the fundamental group of a graph with one vertex, based at the vertex, is a free group with one generator for each edge of the graph. Hence show that any free group occurs as the fundamental group of some graph. *[We have* not *required that a graph have finitely many edges.]*

    (c) Deduce that every subgroup of a free group is free.