Algebraic Topology, Examples 1

Michaelmas 2018

1. Let $a : S^n \to S^n$ be the antipodal map, $a(x) = -x$. Show that a is homotopic to the identity map when n is odd. [Try $n = 1$ first.]

2. Let $f : S^1 \to S^1$ be a map which is not homotopic to the identity map. Show that there exists an $x \in S^1$ such that $f(x) = x$, and a $y \in S^1$ so that $f(y) = -y$.

3. Suppose that $f : X \to Y$ is a map for which there exist maps $g, h : Y \to X$ such that $g \circ f \simeq \mathrm{Id}_X$ and $f \circ h \simeq \mathrm{Id}_Y$. Show that f, g, and h are homotopy equivalences.

4. Show that a retract of a contractible space is contractible.

5. Show that if a space X deformation retracts to a point $x_0 \in X$, then for every open neighbourhood $x_0 \in U$ there exists a smaller open neighbourhood $x_0 \in V \subset U$ such that the inclusion $(V, x_0) \hookrightarrow (U, x_0)$ is based homotopic to the constant map.

6. Construct a space which contains both the annulus $S^1 \times I$ and the Möbius band as deformation retracts.

7. For $m < n$, consider S^m as a subspace of S^n given by

$$\{(x_1, x_2, \ldots, x_{m+1}, 0, \ldots, 0) \mid \sum x_i^2 = 1\}.$$

Show that the complement $S^n - S^m$ is homotopy equivalent to S^{n-m-1}.

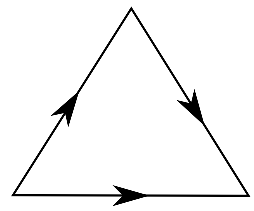
8. A space is called *locally path connected* if for every point $x \in X$ and every neighbourhood $U \ni x$, there exists a smaller neighbourhood V of x (i.e. $x \in V \subset U$) which is path connected. Show that a locally path connected space which is connected is also path connected.

9. For a map $f : S^{n-1} \to X$ we define the *space obtained by attaching an n-cell to X along f* to be the quotient space

$$X \cup_f D^n := (X \amalg D^n)/\sim$$

where $\sim$ is the smallest equivalence relation containing $b \sim f(b)$ for every $b \in S^{n-1} \subset D^n$. Show that if $f, f' : S^{n-1} \to X$ are homotopic maps then $X \cup_f D^n \simeq X \cup_{f'} D^n$.

10. The *dunce cap* is the space obtained from a solid triangle by gluing the edges together as shown.



Show that this space is contractible. [Hint: use the previous question.]

11. Show that the Möbius band does not retract onto its boundary.

12. For based spaces (X, x_0) and (Y, y_0) show there is an isomorphism

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

13. Construct a covering map $\pi : \mathbb{R}^2 \to K$ of the Klein bottle, and hence show that $\pi_1(K, k_0)$ is isomorphic to the group G with elements $(m, n) \in \mathbb{Z}^2$ and group operation

$$(m, n) * (p, q) = (m + (-1)^n \cdot p, n + q).$$

Show that K has a covering space homeomorphic to the torus $S^1 \times S^1$, but that the torus does not have a covering space homeomorphic to K.

14.* A *topological group* consists of a set G equipped with both a topology and a group structure, so that the inversion map $i : G \to G$ (that sends $g \mapsto g^{-1}$) and the multiplication map $m : G \times G \to G$ (that sends $(g, h) \mapsto gh$) are continuous. (Here, $G \times G$ is equipped with the product topology.)

Let G be a path-connected, locally-path-connected topological group, and $p : \widehat{G} \to G$ be a path-connected covering space. Let e be the identity of G and $\epsilon \in p^{-1}(e)$.

(i) Show that $\widehat{G}$ has a unique structure of a topological group with unit ϵ so that p is a continuous homomorphism.

(ii) Show that $\mathrm{Ker}(p) \subset \widehat{G}$ lies in the centre of $\widehat{G}$.

(iii) Show that $SO(3)$, the group of rotations of $\mathbb{R}^3$ (or equivalently of orthogonal 3×3 matrices of determinant 1), is homeomorphic to the projective space $\mathbb{RP}^3$.

(iv) Together, (i) and (iii) give a universal cover $\widetilde{SO(3)}$ homeomorphic to S^3. Identify this group with a well-known matrix group.