

# Algebraic Topology, Examples 4

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Questions marked by \* are optional.

1. Show that if  $n \neq m$  then  $\mathbb{R}^n$  and  $\mathbb{R}^m$  are not homeomorphic.
2. For each of the following exact sequences of abelian groups and homomorphisms say as much as possible about the unknown group  $G$  and homomorphism  $\alpha$ .
  - $0 \rightarrow \mathbb{Z}/2 \rightarrow G \rightarrow \mathbb{Z} \rightarrow 0$ ,
  - $0 \rightarrow \mathbb{Z}/2 \rightarrow G \rightarrow \mathbb{Z}/2 \rightarrow 0$ ,
  - $0 \rightarrow G \rightarrow \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \rightarrow \mathbb{Z}/2 \rightarrow 0$ ,
  - $0 \rightarrow \mathbb{Z}/3 \rightarrow G \rightarrow \mathbb{Z}/2 \rightarrow \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \rightarrow 0$ .
3. Consider a commutative diagram

$$\begin{array}{ccccccccc}
 A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & A_4 & \xrightarrow{f_4} & A_5 \\
 \downarrow h_1 & & \downarrow h_2 & & \downarrow h_3 & & \downarrow h_4 & & \downarrow h_5 \\
 B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & B_3 & \xrightarrow{g_3} & B_4 & \xrightarrow{g_4} & B_5
 \end{array}$$

in which the rows are exact and each square commutes. If  $h_1$ ,  $h_2$ ,  $h_4$ , and  $h_5$  are isomorphisms, show that  $h_3$  is too.

4. For triangulated surfaces  $X$  and  $Y$ , let  $X \# Y$  be the surface obtained by cutting out a 2-simplex from both  $X$  and  $Y$  and then gluing together the two copies of  $\partial\Delta^2$  formed.
  - Use the Mayer–Vietoris sequence to compute the homology of  $\Sigma_g \# \Sigma_h$ , and deduce that it is homeomorphic to  $\Sigma_{g+h}$ .
  - Use the Mayer–Vietoris sequence to compute the homology of  $\Sigma_g \# S_n$ , and hence deduce that it is homeomorphic to  $S_{n+2g}$ .

[Recall that  $\Sigma_g$  denotes the orientable surface of genus  $g$  and that  $S_n$  denotes the non-orientable surface of genus  $n$ .]

5. The *rank* of a finitely generated abelian group  $A$  is the unique integer  $r$  such that  $A \cong \mathbb{Z}^r \oplus A'$  for finite  $A'$ . The *Euler characteristic* of a simplicial complex  $K$  is defined to be

$$\chi(K) = \sum_{i \geq 0} (-1)^i \text{rank } H_i(K).$$

(a) Prove that  $\chi(K) = \sum_{i \geq 0} (-1)^i \text{rank } C_i(K)$ , where  $C_i(K)$  are the chain groups of  $K$ .

(b) Deduce that  $\chi(K) = \sum_{i \geq 0} (-1)^i \#\{i\text{-simplices of } K\}$ .

6. Let  $p : \tilde{X} \rightarrow X$  be a finite-sheeted covering space, and  $h : |K| \rightarrow X$  a triangulation. Show that there is an  $r \geq 1$  and triangulation  $g : |L| \rightarrow \tilde{X}$  so that the composition  $h^{-1} \circ p \circ g : |L| \rightarrow |K^{(r)}|$  is a simplicial map. If  $p$  has  $n$  sheets, show that  $\chi(\tilde{X}) = n \cdot \chi(X)$ . Hence show that  $\Sigma_g$  is a covering space of  $\Sigma_h$  if and only if  $\frac{1-g}{1-h}$  is an integer.

[Hint: If  $g = 1 + k \cdot (h - 1)$ , show that  $\mathbb{Z}/k$  acts freely and properly discontinuously on a particular orientable surface of genus  $g$ , and identify the quotient.]

7. Let  $A$  be a  $2 \times 2$  matrix with entries in  $\mathbb{Z}$ . Show that the linear map  $A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  preserves the equivalence relation  $(a, b) \sim (a', b') \iff (a - a', b - b') \in \mathbb{Z}^2$ , and so induces a continuous map  $f_A$  from the torus  $T$  to itself. Compute the effect of the continuous map  $f_A$  on the homology of  $T$ .

8. \* Let  $K$  be a simplicial complex in  $\mathbb{R}^m$ , and consider this as lying inside  $\mathbb{R}^{m+1}$  as the vectors of the form  $(x_1, \dots, x_n, 0)$ . Let  $e_+ = (0, \dots, 0, 1) \in \mathbb{R}^{m+1}$  and  $e_- = (0, \dots, 0, -1) \in \mathbb{R}^{m+1}$ . The *suspension* of  $K$  is the simplicial complex in  $\mathbb{R}^{m+1}$

$$SK := K \cup \{\langle v_0, \dots, v_n, e_+ \rangle, \langle v_0, \dots, v_n, e_- \rangle \mid \langle v_0, \dots, v_n \rangle \in K\}.$$

(i) Show that  $SK$  is a simplicial complex, and that if  $|K| \cong S^n$  then  $|SK| \cong S^{n+1}$ .

(ii) Using the Mayer–Vietoris sequence, show that if  $K$  is connected then  $H_0(SK) \cong \mathbb{Z}$ ,  $H_1(SK) = 0$ , and  $H_i(SK) \cong H_{i-1}(K)$  if  $i \geq 2$ .

(iii) If  $f : K \rightarrow K$  is a simplicial map, let  $Sf : SK \rightarrow SK$  be the unique simplicial map which agrees with  $f$  on the subcomplex  $K$  and fixes the points  $e_+$  and  $e_-$ . Show that under the isomorphism in (ii), the maps  $f_*$  and  $Sf_*$  agree. [It may help to describe the isomorphism in (ii) at the level of chains.]

(iv) Deduce that for every  $n \geq 1$  and  $d \in \mathbb{Z}$  there is a map  $f : S^n \rightarrow S^n$  so that  $f_*$  induces multiplication by  $d$  on  $H_n(S^n) \cong \mathbb{Z}$ .