

Algebraic Topology, Examples 2

Michaelmas 2015

1. Let X be a Hausdorff space, and G a group acting on X by homeomorphisms, *freely* (i.e. if $g \in G$ satisfies $g \cdot x = x$ for some $x \in X$, then $g = e$) and *properly discontinuously* (i.e. each $x \in X$ has an open neighbourhood $U \ni x$ such that $\{g \in G \mid g(U) \cap U \neq \emptyset\}$ is finite).

1. Show that the quotient map $X \rightarrow X/G$ is a covering map.
2. Deduce that if X is simply-connected and locally path-connected then for any point $[x] \in X/G$ we have an isomorphism of groups $\pi_1(X/G, [x]) \cong G$.
3. Hence show that for $n \geq 2$ odd and any $m \geq 2$ there is a space X with fundamental group \mathbb{Z}/m and universal cover S^n . [Consider S^n as the unit sphere in \mathbb{C}^k .]

2. Show that the Klein bottle has a cell structure with a single 0-cell, two 1-cells, and a single 2-cell. Deduce that its fundamental group has a presentation $\langle a, b \mid baba^{-1} \rangle$, and show this is isomorphic to the group in Q13 of Sheet 1.

3. Show that the inclusion $i : (S^1 \times \{1\}) \cup (\{1\} \times S^1) \hookrightarrow S^1 \times S^1$ does not admit a retraction. [Where $S^1 \subset \mathbb{C}$ is the elements of unit modulus, containing 1.]

4. A *graph* G is a space obtained by starting with a set $E(G)$ of copies of the interval I and an equivalence relation \sim on $E(G) \times \{0, 1\}$, and forming the quotient space of $E(G) \times I$ by the minimal equivalence relation containing \sim . (More practically, it is a space obtained from a collection of copies of I by gluing their ends together.) The *vertices* are the equivalence classes represented by the ends of the intervals.

A *tree* is a graph which is contractible. A tree T inside a graph G is *maximal* if no strictly larger subgraph is a tree.

- (a) If $T \subset G$ is a tree, show that the quotient map $G \rightarrow G/T$ is a homotopy equivalence, and that G/T is again a graph. Hence show that every connected graph is homotopy equivalent to a graph with a single vertex.

[Hint: You may wish to apply Proposition 0.17 from Hatcher].

(b) Show that the fundamental group of a graph with one vertex, based at the vertex, is a free group with one generator for each edge of the graph. Hence show that any free group occurs as the fundamental group of some graph. [We have *not* required that a graph have finitely many edges.]

(c) Show that a covering space of a graph is again a graph, and deduce that a subgroup of a free group is again a free group.

5. Consider $X = S^1 \vee S^1$ with basepoint x_0 the wedge point, which has $\pi_1(X, x_0) = \langle a, b \rangle$ where a and b are given by the two characteristic loops. Describe covering spaces associated to

1. $\langle \langle a \rangle \rangle$, the normal subgroup generated by a ,
2. $\langle a \rangle$, the subgroup generated by a ,
3. the kernel of the homomorphism $\phi : \langle a, b \rangle \rightarrow \mathbb{Z}/4$ given by $\phi(a) = [1]$ and $\phi(b) = [3] = [-1]$.

Show that the free group on two letters contains a copy of itself as a proper subgroup.

6. Consider the 2-dimensional cell complex Y obtained from X in the previous question by attaching 2-cells along loops in the homotopy classes a^2 and b^2 , so that

$$\pi_1(Y, x_0) \cong \langle a, b \mid a^2, b^2 \rangle.$$

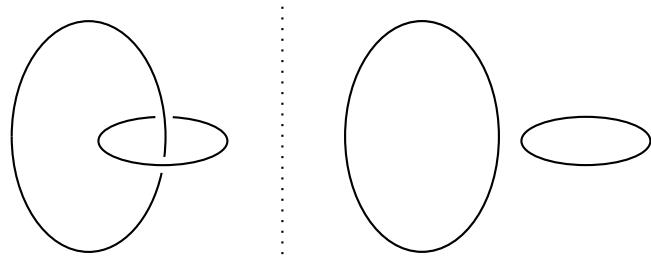
1. Construct (in pictures) the covering space of Y corresponding to the subgroup $\langle a \mid a^2 \rangle$.
2. Construct (in pictures) the covering space of Y corresponding to the kernel of the homomorphism $\phi : \langle a, b \mid a^2, b^2 \rangle \rightarrow \mathbb{Z}/2$ given by $\phi(a) = 1$ and $\phi(b) = 0$. Hence show that $\text{Ker}(\phi)$ is isomorphic to $\langle a, b \mid a^2, b^2 \rangle$.

7. Show that the groups

$$G = \langle a, b \mid a^3b^{-2} \rangle \quad \text{and} \quad H = \langle x, y \mid xyxy^{-1}x^{-1}y^{-1} \rangle$$

are isomorphic. Show that this group is non-abelian and infinite. [Construct surjective homomorphisms to S_3 and \mathbb{Z} .]

8. Consider the following configurations of pairs of circles in S^3 (we have drawn them in \mathbb{R}^3 ; add a point at infinity).



By computing the fundamental groups of the complements of the circles, show there is no homeomorphism of S^3 taking one configuration to the other.