Algebraic Topology Examples 4

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Starred questions are not necessarily harder than the unstarred ones (which are, in any case, not all equally difficult), but they go beyond what you need to know for the course. Comments and corrections are welcome, and should be sent to `ptj@dpmms.cam.ac.uk`.

1. For each of the following exact sequences of abelian groups and homomorphisms, say as much as you can about the unknown group G and/or the unknown homomorphism α:
(i) $0 \to \mathbb{Z}/2\mathbb{Z} \to G \to \mathbb{Z} \to 0$;
(ii) $0 \to \mathbb{Z} \to G \to \mathbb{Z}/2\mathbb{Z} \to 0$;
(iii) $0 \to \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \to 0$;
(iv) $0 \to G \to \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$;
(v) $0 \to \mathbb{Z}/3\mathbb{Z} \to G \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \to 0$.

2. Use the Mayer–Vietoris theorem to calculate the homology groups of the following spaces. [You may assume that suitable triangulations exist in each case.]
(i) The Klein bottle K, regarded as the space obtained by glueing together two copies of $S^1 \times I$.
(ii) The space X obtained by removing the interior of a small disc from a torus. [Recall question 5 on sheet 1.]
(iii) The space Y obtained from the space X of part (ii) and a Möbius band M by identifying the boundary of M with the edge of the 'hole' in X.

3. By restricting the (evident) homeomorphism $B^{r+s+2} \cong B^{r+1} \times B^{s+1}$ to the boundaries of these two spaces, and assuming the existence of suitable triangulations, show that we can triangulate S^{r+s+1} as the union of two subcomplexes L and M, where $|L| \simeq S^r$, $|M| \simeq S^s$ and $|L \cap M| \cong S^r \times S^s$. Use this to calculate the homology groups of $S^r \times S^s$ for $r, s \geq 1$. [Distinguish between the cases $r = s$ and $r \neq s$.]

4. Recall the definition of the suspension SK of a simplicial complex K from question 7 on sheet 3.
(i) Use the Mayer–Vietoris theorem to show that $H_r(SK) \cong H_{r-1}(K)$ for $r \geq 2$, and that $H_1(SK) = 0$ if K is connected.
(ii) Let $f \colon K \to K$ be a simplicial map, and let $\widetilde{f} \colon SK \to SK$ be the unique extension of f to a simplicial map which interchanges the two vertices $\mathbf{v}_{\pm}$. Show that, if we identify $H_r(SK)$ with $H_{r-1}(K)$, then $\widetilde{f}_* \colon H_r(SK) \to H_r(SK)$ sends a homology class c to $-f_*(c)$.
(iii) Deduce that if $a \colon S^n \to S^n$ is the antipodal map, then $a_* \colon H_n(S^n) \to H_n(S^n)$ is multiplication by $(-1)^{n+1}$. [Compare question 1 on sheet 1.]

5. Suppose that a simplicial complex K is the union of subcomplexes L and M, and that P is the union of subcomplexes Q and R. Suppose further that $f \colon K \to P$ is a simplicial map which maps L into Q and M into R. Show that there is a commutative diagram

$$\begin{array}{ccccccccc}
\cdots & H_r(L \cap M) & \longrightarrow & H_r(L) \oplus H_r(M) & \longrightarrow & H_r(K) & \longrightarrow & H_{r-1}(L \cap M) & \cdots \\
 & \Big\downarrow f_* & & \Big\downarrow f_* & & \Big\downarrow f_* & & \Big\downarrow f_* & \\
\cdots & H_r(Q \cap R) & \longrightarrow & H_r(Q) \oplus H_r(R) & \longrightarrow & H_r(P) & \longrightarrow & H_{r-1}(Q \cap R) & \cdots
\end{array}$$

in which the rows are Mayer–Vietoris sequences.

6. By considering S^n as the union of the subsets given by the inequalities $|x_{n+1}| \le \frac{1}{2}$ and $|x_{n+1}| \ge \frac{1}{2}$, and using the results of questions 4 and 5, show that the homology groups of real projective space $\mathbb{R}P^n$ are given by

$$\begin{aligned} H_r(\mathbb{R}P^n) &\cong \mathbb{Z} && \text{if } r = 0, \text{ or if } r = n \text{ and } n \text{ is odd} \\ &\cong \mathbb{Z}/2\mathbb{Z} && \text{if } r \text{ is odd and } 0 < r < n \\ &= 0 && \text{if } r > n, \text{ or if } 0 < r \le n \text{ and } r \text{ is even.} \end{aligned}$$

[You may assume the existence of suitable triangulations.]

7. Calculate the homology groups of the *lens space* L_q obtained from B^2 (considered as the unit disc in $\mathbb{C}$) by identifying points z_1, z_2 on its boundary whenever $z_1^q = z_2^q$. Using the result of question 4(i), deduce that for any finite sequence $G_1, G_2, \ldots, G_n$ of finitely-generated abelian groups there exists a polyhedron X with $H_0(X) \cong \mathbb{Z}$, $H_i(X) \cong G_i$ for $1 \le i \le n$ and $H_i(X) = 0$ for $i > n$. [Compare question 7 on sheet 2; you may assume the structure theorem which says that any finitely-generated abelian group is isomorphic to a finite direct sum of (finite or infinite) cyclic groups.]

8. (i) Let A be a 2×2 matrix with integer entries. Show that the linear map $\mathbb{R}^2 \to \mathbb{R}^2$ represented by A respects the equivalence relation $\sim$ on $\mathbb{R}^2$ given by $(x, y) \sim (z, w)$ iff $x - z$ and $y - w$ are integers, and deduce that it induces a continuous map f_A from the torus T to itself. Calculate the effect of f_A on the homology groups of T, and show in particular that f_A is a homeomorphism if and only if it induces an isomorphism $H_2(T) \to H_2(T)$. [It can be shown that every continuous map $T \to T$ is homotopic to f_A for some A.]

(ii) For which matrices A do there exist continuous maps $T \to T$ homotopic to f_A with no fixed points?

9. Show that if $f\colon S^n \to S^n$ is a continuous map without fixed points, then $f_*\colon H_n(S^n) \to H_n(S^n)$ is multiplication by $(-1)^{n-1}$. [Note that this provides a simpler proof of the result of question 4(iii).] Deduce that if n is even and $p\colon S^n \to X$ is a nontrivial covering projection then $\Pi_1(X) \cong \mathbb{Z}/2\mathbb{Z}$. [Compare question 4 on sheet 2. Remarkably, such an X need not be homeomorphic to $\mathbb{R}P^n$; there is a counterexample with $n = 4$.]

10. (i) Let $p\colon Y \to X$ be a k-fold covering projection, and suppose X is triangulable. Show that we can choose a triangulation of X in which each simplex is evenly covered by p, and deduce that $\chi(Y) = k \,.\, \chi(X)$.

(ii) Show that each non-orientable triangulable 2-manifold has a double covering by an orientable 2-manifold.

***11**. By a *knot* in S^3, we mean an embedding $f\colon S^1 \to S^3$ — that is, a homeomorphism from S^1 to a subspace of S^3. We say the knot is *tame* if f can be 'thickened up' to an embedding $\widetilde{f}\colon S^1 \times B^2 \to S^3$, with $f = \widetilde{f}|_{S^1 \times \{0\}}$. Assuming (as always!) the existence of suitable triangulations, show that if f is a tame knot, then the space obtained by removing the interior of the image of $\widetilde{f}$ from S^3 has the same homology groups as S^1. [Thus homology groups, unlike the fundamental group (see question 13 on sheet 2), are of no use for distinguishing between different knots.]

***12**. Consider the three stainless steel sculptures along the Clarkson Road front of the Isaac Newton Institute.

(i) Are they topologically equivalent? (That is, are there homeomorphisms $\mathbb{R}^3 \to \mathbb{R}^3$ mapping each of them onto the other two?)

(ii) Prove that the sculptor could not have made a sculpture topologically equivalent to any of them using three (circular) solid tori. [Method: given three circles in $\mathbb{R}^3$, let O be the point of intersection of the three planes in which they lie. Observe that O must lie inside each of the circles if they are linked as in the sculptures, and consider the three quantities $r_i^2 - OC_i^2$, where r_i is the radius of the ith circle and C_i its centre.]