Algebraic Topology Examples 2

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Starred questions are not necessarily harder than the unstarred ones (which are, in any case, not all equally difficult), but they go beyond what you need to know for the course. Comments and corrections are welcome, and should be sent to `ptj@dpmms.cam.ac.uk`.

1. Show that the covering projection $\mathbb{R}^2 \to K$ from question 11 on sheet 1 may be factored as $\mathbb{R}^2 \to T \to K$, where T is the 2-dimensional torus $S^1 \times S^1$. Identify the subgroup of index 2 in $\Pi_1(K)$, isomorphic to $\mathbb{Z} \times \mathbb{Z}$, corresponding to the covering $T \to K$.

2. Show that the free group F_2 on two generators has exactly three subgroups of index 2 [hint: consider homomorphisms $F_2 \to \mathbb{Z}/2\mathbb{Z}$]. Draw pictures of the three corresponding double coverings of $S^1 \vee S^1$, and calculate their fundamental groups.

3. Let X be the subspace of $\mathbb{R}^2$ which is the union of the x-axis with $\bigcup_{n\in\mathbb{Z}} C_n$, where C_n is the circle with centre $(n, \frac{1}{3})$ and radius $\frac{1}{3}$. Construct a covering projection $X \to S^1 \vee S^1$. Show that X is homotopy equivalent to a countably-infinite wedge union of circles, and deduce that the free group on two generators contains a subgroup which is free on countably many generators.

4. Let X be a Hausdorff topological space, and let G be a finite subgroup of the group of all homeomorphisms $X \to X$, such that no member of G other than the identity has a fixed point. Let X/G denote the set of G-orbits, topologized as a quotient space of X. Show that the quotient map $X \to X/G$ is a covering projection, and deduce that if X is simply connected and locally path-connected then $\Pi_1(X/G) \cong G$. Hence show that, for any odd $n > 1$ and any $m > 1$, there is a quotient space of S^n with fundamental group $\mathbb{Z}/m\mathbb{Z}$. [Hint: regard $\mathbb{R}^{2k}$ as $\mathbb{C}^k$. In contrast, we'll see later that for even n, the only group which can act on S^n so that no non-identity element has a fixed point is the cyclic group of order 2.]

5. Let X be the subspace

$$\{(x, \sin\ \pi/x) \mid 0 \le x \le 1\} \cup \{(0, y) \mid -1 \le y \le 1\}$$

of $\mathbb{R}^2$, and let Y be the union of X with the three line segments $\{(0, y) \mid 1 \le y \le 2\}$, $\{(x, 2) \mid 0 \le x \le 1\}$ and $\{(1, y) \mid 0 \le y \le 2\}$.

(i) Show that X is connected but not path-connected (if you didn't already do this in Metric and Topological Spaces).

(ii) Show that Y is simply connected but not locally path-connected.

(iii) Show that there is a double covering $p\colon Z \to Y$ where Z is connected (and thus not homeomorphic to $Y \times \{1, 2\}$).

6. (i) Let H be the Hawaiian earring $\bigcup_{n=1}^{\infty} C_n \subseteq \mathbb{R}^2$, where C_n is the circle with centre $(0, -\frac{1}{n})$ and radius $\frac{1}{n}$. Show that $\Pi_1(H, (0, 0))$ is uncountable, and deduce that it is not finitely presented. [In showing that you have constructed uncountably many distinct elements of $\Pi_1(H)$, you may find it helpful to consider the continuous maps $H \to S^1$ collapsing all but one of the circles in H to a point, and wrapping the remaining one around S^1.]

*(ii) Let H' be the reflection of H in the x-axis. Is $\Pi_1(H \cup H', (0, 0))$ isomorphic to the free product of two copies of $\Pi_1(H, (0, 0))$?

*(iii) Now regard H and H' as embedded in the plane $\{(x, y, z) \mid z = 0\} \subseteq \mathbb{R}^3$; let C be the cone on H with vertex $(0, 0, 1)$, and C' the cone on H' with vertex $(0, 0, -1)$. Is $C \cup C'$ simply connected?

7. Let $f\colon S^1 \to X$ be a continuous map, and consider the space $Y = X \cup_f B^2$, defined as in question 9 on sheet 1. Let $x = f(1)$; show that $\Pi_1(Y, x) \cong \Pi_1(X, x)/N$, where N is the normal subgroup generated by $f_*(g)$ for a generator g of $\Pi_1(S^1)$. Deduce that, for any finitely presented group G, there is a compact path-connected space Z with $\Pi_1(Z) \cong G$.

8. Show that the Klein bottle K may be described as $(S^1 \vee S^1) \cup_f B^2$ for a suitable map $f\colon S^1 \to S^1 \vee S^1$. Use question 7 to give a presentation of $\Pi_1(K)$ with two generators and one relation, and verify directly that this group is isomorphic to the one described in question 11 on sheet 1.

9. Show that the finitely presented groups

$$G = \langle a, b \mid a^3 = b^2 \rangle \qquad \text{and} \qquad H = \langle x, y \mid xyx = yxy \rangle$$

are isomorphic. Show also that this group is non-abelian and infinite. [Hint: find surjective homomorphisms to the symmetric group S_3 and to $\mathbb{Z}$.]

10. Complex projective space $\mathbb{C}P^n$ is the quotient of $\mathbb{C}^{n+1} \setminus \{0\}$ by the equivalence relation which identifies $\mathbf{x}$ and $\mathbf{y}$ if $\mathbf{x} = t\mathbf{y}$ for some (complex) scalar t. Show that

(i) there is a quotient map $h_n\colon S^{2n+1} \to \mathbb{C}P^n$ such that the inverse image of each point is a copy of S^1;

(ii) for $n > 1$, $\mathbb{C}P^n$ is homeomorphic to $\mathbb{C}P^{n-1} \cup_{h_{n-1}} B^{2n}$;

(iii) $\mathbb{C}P^1$ is homeomorphic to S^2.

Deduce that $\mathbb{C}P^n$ is simply connected for all n.

*__11__. Let G be a connected graph, considered as a topological space in the way that we did for the Cayley graph of F_2 in lectures. Show that there is a simply connected subgraph G' containing all the vertices of G, and deduce that $\Pi_1(G)$ is isomorphic to the free group generated by the edges of $G \setminus G'$. Hence show (generalizing the result of question 3) that any subgroup of a free group is free.

*__12__. (i) Let $G = SU(2)$ be the group of 2×2 unitary matrices with determinant 1. Show that (the underlying space of) G is homeomorphic to S^3.

(ii) Let $H = SO(3)$, the group of 3×3 orthogonal matrices with determinant 1. Show that H is homeomorphic to $\mathbb{R}P^3$. [Hint: first show that the set of 180° rotations is homeomorphic to $\mathbb{R}P^2$.]

(iii) Show that $\{\pm I\}$ is a normal subgroup of $SU(2)$, and that the quotient $SU(2)/\{\pm I\}$ is isomorphic (as well as homeomorphic) to $SO(3)$.

(iv) Show that there is a quotient space X of S^3 such that $\Pi_1(X)$ is a non-abelian group of order 120, whose only nontrivial normal subgroup has order 2. [Use question 4: recall that the group of rotational symmetries of a regular dodecahedron is isomorphic to A_5.]

*__13__. View S^3 as the set $\{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\}$, and let $T \subseteq S^3$ be the set

$$\{(z, w) \mid z^2 = w^3\} \ .$$

(T is called the *trefoil knot*.)

(i) Show that T is homeomorphic to S^1.

(ii) Use the Seifert–Van Kampen theorem to show that $\Pi_1(S^3 \setminus T)$ is isomorphic to the group G in question 9. [Method: let $X = \{(z, w) \mid |z|^2 = |w|^3\}$. Show that $S^3 \setminus X$ is homeomorphic to the disjoint union of two copies of $S^1 \times B^2$, and that $X \setminus T$ is homeomorphic to $S^1 \times (0, 1)$.]

(iii) Let $U = \{(z, 0) \mid |z| = 1\}$ be the *unknot* in S^3. Show that there is no homeomorphism $f\colon S^3 \to S^3$ for which $f(U) = T$. [Recall question 6 on sheet 1.]