Algebraic Topology Examples 1

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Starred questions are not necessarily harder than the unstarred ones (which are, in any case, not all equally difficult), but they go beyond what you need to know for the course. Comments and corrections are welcome, and should be sent to `ptj@dpmms.cam.ac.uk`.

1. Let $a\colon S^n \to S^n$ be the *antipodal map* (defined by $a(\mathbf{x}) = -\mathbf{x}$). Show that a is homotopic to the identity map if n is odd. [Hint: try $n = 1$ first! Later in the course, we'll be able to strengthen 'if' to 'if and only if'.]

2. Which of the capital letters $A, B, C, \ldots, Z$ are contractible? And which are homotopy equivalent to S^1?

3. Let $f\colon X \to Y$ be a continuous map, and suppose we are given (not necessarily equal) continuous maps $g, h\colon Y \rightrightarrows X$ such that $gf \simeq \mathrm{id}_X$ and $fh \simeq \mathrm{id}_Y$. Show that f is a homotopy equivalence.

4. (i) Let Y be the subspace $\{(x, 0) \mid x \in \mathbb{Q}, 0 \le x \le 1\}$ of $\mathbb{R}^2$, and let X be the *cone* on Y with vertex $(0, 1)$, i.e. the set of all points on straight line segments joining points of Y to $(0, 1)$. Show that X is contractible, but that in any homotopy H between the identity map on X and the constant map with value $(0, 0)$, the point $(0, 0)$ must 'move' (i.e. there exists t with $H((0, 0), t) \neq (0, 0)$).

*(ii) The problem in (i) arose because we chose the 'wrong' basepoint for X: if we had chosen $(0, 1)$ instead of $(0, 0)$, all would have been well. Can you find a contractible space Z such that *every* point of Z has to move in the course of a contracting homotopy?

5. Show that the torus minus a point, and the Klein bottle minus a point, are both homotopy equivalent to $S^1 \vee S^1$. [Hint: draw pictures showing how $S^1 \vee S^1$ can be embedded as a deformation retract in each space; do not attempt to write down precise formulae for the homotopies.]

6. Consider S^m embedded in S^n ($m < n$) as the subspace $\{(x_1, x_2, \ldots, x_{m+1}, 0, \ldots, 0) \mid \sum x_i^2 = 1\}$. Show that $S^n \setminus S^m$ is homotopy equivalent to S^{n-m-1}.

7. Let (X, x) and (Y, y) be two based spaces. Prove that $\Pi_1(X \times Y, (x, y)) \cong \Pi_1(X, x) \times \Pi_1(Y, y)$.

8. (i) Let A be a set equipped with two binary operations $\cdot$ and $*$, having a common (two-sided) identity element c and satisfying the 'interchange law'

$$(p \cdot q) * (r \cdot s) = (p * r) \cdot (q * s)$$

which says that each of the operations is a 'homomorphism' relative to the other. Show that the two operations coincide, and that they are (it is?) associative and commutative. [Hint: make appropriate substitutions in the interchange law. This piece of pure algebra is known as the *Eckmann–Hilton argument*: it has many applications besides the two described below.]

(ii) Let X be a space equipped with a continuous binary operation $m\colon X \times X \to X$ having a two-sided identity element e. Use part (i) and the previous question to show that $\Pi_1(X, e)$ is abelian. [Familiar examples of such spaces include topological groups; but the existence of inverses, and even the associativity of multiplication, are not needed for this result.]

*(iii) The *second homotopy group* $\Pi_2(X, x)$ of a pointed space (X, x) has elements which are homotopy classes of continuous maps from the unit square I^2 to X which map the boundary ∂I^2 to x (the homotopies between such maps being required to fix ∂I^2). Show that there are two possible ways ('horizontal' and 'vertical') of composing two such '2-dimensional loops', and deduce that $\Pi_2(X, x)$ is an abelian group. [For $n > 2$, $\Pi_n(X, x)$ is defined similarly using the unit n-cube I^n; it too is always abelian.]

9. Recall that, given a continuous map $f\colon S^{n-1} \to X$, we write $X \cup_f B^n$ for the space obtained by glueing an n-ball to X along f, i.e. the quotient of the disjoint union of X and B^n by the smallest equivalence relation which identifies $\mathbf{x}$ with $f(\mathbf{x})$ for each $\mathbf{x} \in S^{n-1}$. If f and g are homotopic maps $S^{n-1} \rightrightarrows X$, show that the spaces $X \cup_f B^n$ and $X \cup_g B^n$ are homotopy equivalent.

10. The 'topologist's dunce cap' D is the space obtained from the cone on the circle $\{(x, y, 0) \mid x^2 + y^2 = 1\}$ with vertex $(0, 0, 1)$ by identifying the points $(\cos\ 2\pi t, \sin\ 2\pi t, 0)$ and $(1-t, 0, t)$ for $0 \le t \le 1$. Show that D is contractible. [Hint: use the previous question; it's helpful to 'flatten out' the cone by cutting it along the line $\{(1-t, 0, t) \mid 0 \le t \le 1\}$.]

11. Construct a covering projection $p\colon \mathbb{R}^2 \to K$ where K is the Klein bottle, and use it to show that $\Pi_1(K)$ is isomorphic to the group whose elements are pairs $(m, n) \in \mathbb{Z}^2$, with group operation given by

$$(m, n) * (p, q) = (m + (-1)^n p, n + q) \quad .$$

***12**. Let $p\colon X' \to X$ be a covering projection, and suppose given basepoints x, x' with $p(x') = x$. Show that, for any $n > 1$, p induces an isomorphism $\Pi_n(X', x') \cong \Pi_n(X, x)$ (for the definition of Π_n, see question 8(iii)). Deduce that $\Pi_n(S^1)$ is trivial for all $n > 1$. [Warning: this result does *not* generalize to higher-dimensional spheres: we have $\Pi_n(S^n) \cong \mathbb{Z}$ for all n, but $\Pi_m(S^n)$ can be nontrivial for $m > n$.]

***13**. Let X be an arbitrary metric space, and K a compact metric space. Given two continuous maps $f, g\colon K \rightrightarrows X$, explain why the function $k \mapsto d(f(k), g(k))$ (where d is the metric on X) is bounded and attains its bounds. Show also that

$$\overline{d}(f, g) = \sup\{d(f(k), g(k)) \mid k \in K\}$$

defines a metric on the set $\mathrm{Cts}(K, X)$ of all continuous maps $K \to X$.

Given $H\colon K \times I \to X$, show that H is continuous if and only if the function $\widehat{H}$ defined by $\widehat{H}(t)(k) = H(k, t)$ is a continuous function $I \to \mathrm{Cts}(K, X)$. Deduce that X is simply connected if and only if $\mathrm{Cts}(S^1, X)$ is path-connected.

***14**. A space X is said to be *locally path-connected* (or sometimes *semi-locally path-connected*) if, given any $x \in X$ and any open neighbourhood U of x, there exists a smaller open neighbourhood $V \subseteq U$ such that any two points of V may be joined by a path taking values in U. If X is a metric space, show that this condition is equivalent to saying that the mapping $f\colon \mathrm{Cts}(I, X) \to X$ sending a path u to $u(0)$ is an open map. (Recall that a map $g\colon Y \to Z$ between topological spaces is said to be *open* if $g(U)$ is open in Z whenever U is open in Y.)

At first sight, a more 'natural' definition of local path-connectedness would be to say that every open subset can be written as a union of path-connected open subsets (i.e., the path-connected opens form a base for the topology). Can you find an example of a space which fails to satisfy this condition but satisfies the one in the previous paragraph? [The counterexample from question 4(ii) might be helpful.]