EXAMPLE SHEET 2

1. What is $\pi_1(S^1 \vee S^2, *)$? Draw the universal cover of this space.

2. Prove that if $f, g : S^n \to X$ are homotopic maps, then $X \cup_f B^{n+1}$ and $X \cup_g B^{n+1}$ are homotopy equivalent.

3. Suppose $f : (S^1, *) \to (X, *)$ is a based map ($i.e$ $f(*) = *$), and let $Y = X \cup_f B^2$ be the space obtained by gluing B^2 to X using f.

   (a) Let $\alpha \in \pi_1(X, *)$ be the element represented by f. Show that $\pi_1(Y, *) \cong \pi_1(X)/N$, where N is the normal subgroup generated by α.

   (b) Use (a) to show that any finitely presented group is the fundamental group of some space.

4. Show that the finitely presented groups

$$G = \langle a, b \mid a^3 = b^2 \rangle$$
$$H = \langle x, y \mid xyx = yxy \rangle$$

   are isomorphic. Show that G is a nonabelian group of infinite order. (Hint: find surjective homomorphisms to the symmetric group S_3 and to $\mathbb{Z}$.)

5. Let K be the Klein bottle. Recall that K is constructed from the cylinder $S^1 \times [0, 1]$ by identifying the point $(x, 0)$ with $(-x, 1)$.

   (a) Construct a covering map $p : T^2 \to K$. Describe the group of deck transformations.

   (b) Construct a covering map $p' : \mathbb{R}^2 \to K$. Describe the group of deck transformations.

   (c) Using (b), show that $\pi_1(K)$ is isomorphic to the group G whose elements are pairs $(m, n) \in \mathbb{Z}^2$, with group operation given by

$$(m, n) * (p, q) = (m + (-1)^n p, n + q).$$

   (d) Describe K as $(S^1 \vee S^1) \cup_f B^2$ for some map $f : S^1 \to S^1 \vee S^1$. Use this description to give a presentation of $\pi_1(K)$ with two generators and one relation.

   (e) Show directly that the group given by this presentation is isomorphic to G.

In the next three problems, let $G = \pi_1(S^1 \vee S^1, *)$ be the free group on two generators a and b.

6. Show that there are precisely three nontrivial homomorphisms from G to $\mathbb{Z}/2$. Draw pictures of the cover of $S^1 \vee S^1$ corresponding to each homomorphism. Describe the covering map in each case.

7. Define $f : G \to \mathbb{Z}$ to be the homomorphism with $f(a) = 0$ and $f(b) = 1$, and let X be the corresponding covering space of $S^1 \vee S^1$. Show that X is homotopy equivalent to an infinite wedge of circles. Conclude that the free group on infinitely many generators is isomorphic to a subgroup of the free group on two generators.

8. Draw a diagram of the covering space of $S^1 \vee S^1$ corresponding to the subgroup H of G in each of the following cases:

   (a) H is the subgroup generated by a.

   (b) H is the smallest normal subgroup containing a.

   (c) H is the commutator subgroup of G.

9.* View S^3 as the set $\{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\}$, and let $K \subset S^3$ be the set

$$\{(z, w) \mid z^2 = w^3\}.$$

K is called the *trefoil knot*.

   (a) Show that K is homeomorphic to S^1.

   (b) Use the Seifert-Van Kampen theorem to show that $\pi_1(S^3 - K)$ is isomorphic to the group G in example 4. (Hint: Let $T = \{(z, w) \mid |z|^2 = |w|^3\}$. Show that $S^3 - T$ is homeomorphic to the disjoint union of two copies of $S^1 \times B^2$, and that $T - K$ is homeomorphic to $S^1 \times (0, 1)$.)

   (c) Let $U = \{(z, 0) \mid |z| = 1\}$ be the *unknot* in S^3. Show that there is no homeomorphism $f : S^3 \to S^3$ for which $f(U) = K$.

10.* Let X be a graph. Show that the fundamental group of X is free. (Hint: Start with a maximal tree $T \subset X$.) Conclude that any subgroup of a free group is free.

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