ALGEBRAIC TOPOLOGY **MICHELMAS 2008**

EXAMPLE SHEET 4

1. For each of the following exact sequences of abelian groups, say as much as you can about the unknown group G and/or the unknown homomorphism α.

   (a) $0 \to \mathbb{Z}/2 \to G \to \mathbb{Z} \to 0$
   (b) $0 \to \mathbb{Z} \to G \to \mathbb{Z}/2 \to 0$
   (c) $0 \to \mathbb{Z}/2 \to G \to \mathbb{Z}/2 \to 0$
   (d) $0 \to \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}/2 \to 0$
   (e) $0 \to G \to \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \to \mathbb{Z}/2 \to 0$

2. Use the Mayer-Vietoris sequence to compute the homology of the following spaces. You may assume the existence of appropriate triangulations in each case.

   (a) The Klein bottle. (Decompose as the union of two copies of $S^1 \times [0, 1]$.)
   (b) $T^2 \# \mathbb{RP}^2$. (Decompose as the union of $T^2 - D^2$ and the Mobius strip.)

3. Let X be the space obtained from $T^2 \simeq S^1 \times S^1$ by collapsing $A = S^1 \times 0$ to a point. Show that X is homeomorphic to the space S^2/B, where B is a subset of S^2 consisting of two points. Compute $H_*(X)$ by using

   (a) the exact sequence of the pair (T^2, A)
   (b) the exact sequence of the pair (S^2, B)
   (c) the Mayer-Vietoris sequence.

   You may assume the existence of appropriate triangulations in each case.

4. List all the closed surfaces (orientable or not) and compute their Euler characteristics.

5. Use the Mayer-Vietoris sequence to compute $H_*(S^n \times S^m)$.

6. If X is a simplicial complex, the *suspension* SX of X is $(X \times [0, 1])/\sim$, where $(x, 0) \sim (y, 0)$ and $(x, 1) \sim (y, 1)$ for all $x, y \in X$. (In other words, we collapse $X \times 0$ and $X \times 1$ to two separate points.) If $f : X \to X$, we define $f_S : SX \to SX$ by $f_S(x, t) = (f(x), t)$.

   (a) Show that $SX = CX \cup_X CX$. (For CX, see problem 6 on the third example sheet.)

(b) Show that for $i > 0$, $H_i(X) \cong H_{i+1}(SX)$. If X is connected, show that $H_1(SX) = 0$.

(c) Let $\phi : H_i(X) \to H_{i+1}(SX)$ be the isomorphism of part (b). Show that the following diagram commutes:

$$\begin{array}{ccc} H_i(X) & \xrightarrow{f_*} & H_i(X) \\ \phi \big\downarrow & & \phi \big\downarrow \\ H_{i+1}(X) & \xrightarrow{(f_S)_*} & H_{i+1}(X) \end{array}$$

(It may help to understand ϕ at the chain level.)

(d) Show that for each $d \in \mathbb{Z}$ there is a map $f_d : S^n \to S^n$ of degree d.

7. (a) Let $f : S^n \to S^n$ be given by $f((x_1, x_2, x_3, \ldots, .., x_{n+1})) = (-x_1, x_2, x_3, \ldots, x_{n+1})$. Use problem 6 to show that f has degree -1.

   (b) Let $g : S^n \to S^n$ be the antipodal map: $g(\mathbf{x})) = -\mathbf{x}$. Show that g has degree $(-1)^{n+1}$. Conclude that if n is even, g is not homotopic to the identity map. (Compare problem 1 on the first example sheet.)

8. Suppose Y is a finite simplicial complex, and that $p : X \to Y$ is a covering map. If $p^{-1}(y)$ consists of n points for each $y \in Y$, show that X is a finite simplicial complex and that $\chi(X) = n\chi(Y)$. If Y is an orientable manifold, show that X is orientable and that p has degree n. If Y is a surface of genus 4 and $n = 5$, what is the genus of X?

9. If X is a triangulated 3-manifold, show that $\chi(X) = 0$. (Hint: if V, E, F and G are the numbers of $0, 1, 2$ and 3 simplices, then $2V = 2E - 3F + 4G$.)

10. Suppose $K \subset S^3$ is a *tame knot*, *i.e.* K has a neighborhood U homeomorphic to $S^1 \times D^2$ so that $K = S^1 \times 0$. Use the Mayer-Vietoris sequence to compute $H_*(S^3 - U)$.

11. Suppose $p : S^n \to X$ is a covering map, and let G be the group of deck transformations.

   (a) If n is even, show that $G \cong \mathbb{Z}/2$. (Interestingly, there is a covering map $S^4 \to X$ with $X \not\cong \mathbb{RP}^4$.)

   (b) If n is odd, show there is a covering map $p : S^n \to X$ with deck group Z/p for any $p > 0$. (Hint: view $\mathbb{R}^{2k}$ as $\mathbb{C}^k$.)

   (c)* Show there is a covering map $p : S^3 \to X$ for which $|G| = 120$, and that $H_1(X) = 0$. When $n = 3$, this is the largest possible order of G. The manifold X is called the *Poincaré sphere*. (Hint: A_5 is a subgroup of $SO(3)$.)

J.Rasmussen@dpmms.cam.ac.uk