ALGEBRAIC TOPOLOGY (PART II) EXAMPLE SHEET 2

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(1) Using homotopy theory, prove the fundamental theorem of algebra, that is, if $p \in \mathbb{C}[z]$ is a non-constant one-variable polynomial with complex coefficients, then it has a root in $\mathbb{C}$.

(2) (a) Show that the fundamental group of the complement of a finite set in $\mathbb{R}^2$ is a free group.
(b) Show that the complement of a finite set in $\mathbb{R}^3$ is simply connected.

(3) Show that the group $\{a, b \mid a^3 = 1, b^2 = 1, bab^{-1} = a^2\}$ is isomorphic to the symmetric group S_3.

(4) Let X be a space, $x \in X$, $f : S^1 \to X$ a map such that x is in the image of f, and $Y = X \cup_f B^2$ the space obtained by gluing B^2 to X along f.
(a) Let $[f] \in \pi_1(X, x)$ be the element represented by f. Show that $\pi_1(Y, x)$ is isomorphic to $\pi_1(X, x)/N$, where N is the normal subgroup of $\pi_1(X, x)$ generated by $[f]$.
(b) Use (a) to show that every finitely presented group is the fundamental group of some space.

(5) Let L be the "infinite ladder of circles" given by the subset of $\mathbb{R}^2$ consisting of the union of the circle of radius $1/2$ around each point $(n, 0)$ for $n \in \mathbb{Z}$.
(a) Choose a basepoint for L and show that the fundamental group is a free group on a (countably) infinite number of generators.
(b) Show that the quotient space $L/\mathbb{Z}$ is homeomorphic to $S^1 \vee S^1$ where $\mathbb{Z}$ acts on L via $(x, y) \mapsto (x + n, y)$ for $n \in \mathbb{Z}$.
(c) Conclude that the free group on an infinite number of generators is isomorphic to a subgroup of the free group on two

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generators.

(6) Let X be the numeral 8, viewed as a topological space. In other words, X is the wedge of two circles. Draw pictures of the three (connected) double coverings of X, showing that the fundamental group $\pi_1(X, x)$ has exactly three subgroups of index 2.

(7) Prove the Borsuk-Ulam theorem in dimension 2: prove that there is no map $f \colon S^2 \to S^1$ such that $f(-x) = -f(x)$ for every $x \in S^2$. Deduce that S^2 is not homeomorphic to any subset of $\mathbb{R}^2$.

(8) Use the simplicial approximation theorem to show:
(a) If X and Y are compact triangulable spaces, then there are at most countably many homotopy classes of maps from X to Y.
(b) If $m < n$, then every map $S^m \to S^n$ is homotopic to a constant map.

(9) Let K be a simplicial complex.
(a) Show that if $|K|$ is connected, then any two vertices in K can be connected by a sequence of edges in K.
(b) Let K_2 be the 2-skeleton of K, the subcomplex containing all vertices, 1-simplices, and 2-simplices. For a vertex a, show that $\pi_1(K_2, a) \to \pi_1(K, a)$ is an isomorphism.

(10) For each of the following exact sequences of abelian groups, say what you can about the unknown group A and/or the unknown homomorphism α.
(a) $0 \to \mathbb{Z}/2 \to A \to \mathbb{Z} \to 0$
(b) $0 \to \mathbb{Z} \to A \to \mathbb{Z}/2 \to 0$
(c) $0 \to \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}/2 \to 0$
(d) $0 \to A \to \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \to \mathbb{Z}/2 \to 0$
(e) $0 \to \mathbb{Z}/3 \to A \to \mathbb{Z}/2 \to \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \to 0$

(11) The Five Lemma

Consider the following commutative diagram of abelian groups, where the rows are exact.

$$\begin{array}{ccccccccc} A & \longrightarrow & B & \xrightarrow{f} & C & \longrightarrow & D & \longrightarrow & E \\ \downarrow & & \downarrow & & \gamma\downarrow & & \downarrow & & \downarrow \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \end{array}$$

Suppose that the maps $A \to A'$, $B \to B'$, $D \to D'$, and $E \to E'$ are isomorphisms. Show that the middle map $\gamma : C \to C'$ must be an isomorphism, as follows.

(a) First show that $C \to C'$ is injective: Take an element x in C which maps to 0 in C'. (1) Show that x maps to 0 in D and hence that x is the image of some $y \in B$. (2) Show that y is in the image of A and conclude that $x = 0$.

(b) Now that that $C \to C'$ is onto. Take an element $x' \in C'$ and show that it is in the image of C as follows. (1) Show that there is an element $z \in C$ such that $\gamma(z)$ and x' in C' have the same image in D'. (2) Show that there is an element $y \in B$ whose image in B' maps to $x' - \gamma(z)$. Conclude that $\gamma(z + f(y)) = x'$.