Examples sheet 1 for Part IIB Algebraic Topology

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(1) Show that a connected manifold is path-connected. (For the purpose of this problem, we can define an n-dimensional manifold to be a topological space such that every point has an open neighbourhood homeomorphic to an open subset of Euclidean space $\mathbf{R}^n$.)

(2) Let $f : S^1 \to S^1$ be a map which is not homotopic to the identity map. Show that $f(x) = -x$ for some $x \in S^1$.

(3) Let CX be the cone on a space X. By definition, this is the identification space (also called a quotient space) of $X \times [0, 1]$ obtained by identifying all the points $(x, 0)$ for $x \in X$ to the same point. Show that CX is contractible.

(4) Show that the Möbius strip and the cylinder $S^1 \times [0, 1]$ both have fundamental group isomorphic to $\mathbf{Z}$. (We can define the Möbius strip as the space made from the square $[0, 1] \times [0, 1]$ by identifying the point $(0, t)$ with $(1, 1 - t)$ for all t, although it is probably more useful to draw a picture.)

(5) Classify the capital letters $A, B, \ldots, Z$ up to homeomorphism and also up to homotopy type. Of course, it depends on how you write them.

(6) Let X be the numeral 8, viewed as a topological space. In other words, X is the wedge of two circles. Draw pictures of the three (connected) double coverings of X, showing that the fundamental group $\pi_1(X, x)$ has exactly three subgroups of index 2.

(7) Show that the fundamental group of the product of two spaces is the product of their fundamental groups.

(8) Show that the fundamental group of real projective space $\mathbf{RP}^n$, $n \geq 2$, is generated by the image of a great-circle path in S^n from the north to the south pole.

(9) For a map $f : S^1 \to X$, let $X \cup_f D^2$ be the identification space obtained from the disjoint union of X and the 2-disc D^2 by identifying each point x on the boundary S^1 of D^2 to the point $f(x)$ in X. Show that if $f, g : S^1 \to X$ are homotopic maps, then the spaces $X \cup_f D^2$ and $X \cup_g D^2$ are homotopy equivalent.