Algebraic Topology 2004 — Example Sheet 4

1. For each of the following exact sequences of abelian groups, say what you can about the unknown group A and/or the unknown homomorphism α.

   (a) $0 \to \mathbb{Z}/2 \to A \to \mathbb{Z} \to 0$

   (b) $0 \to \mathbb{Z} \to A \to \mathbb{Z}/2 \to 0$

   (c) $0 \to \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}/2 \to 0$

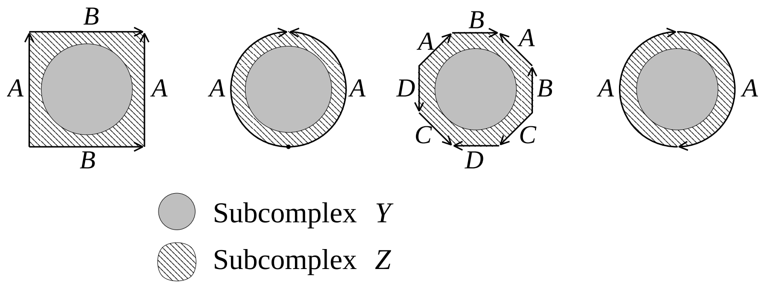
   (d) $0 \to A \to \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \to \mathbb{Z}/2 \to 0$

   (e) $0 \to \mathbb{Z}/3 \to A \to \mathbb{Z}/2 \to \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \to 0$

2. Let X and Y be triangulable spaces, and choose basepoints $x \in X$, $y \in Y$. Show that $\tilde{H}_*(X \vee Y) \cong \tilde{H}_*(X) \oplus \tilde{H}_*(Y)$. Show that for a wedge of n circles,

$$H_*(S^1 \vee \cdots \vee S^1) = \begin{cases} \mathbb{Z} & \text{if } * = 0 \\ \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} & \text{if } * = 1 \qquad (n \text{ summands}) \\ 0 & \text{otherwise.} \end{cases}$$

3. For X the torus, the sphere, the two-holed torus, and the projective plane, assume there is a triangulation $X = Y \cup Z$ as pictured (with $Y \cap Z$ homeomorphic to the circle), and use the Mayer-Vietoris sequence to compute $H_*(X)$.



4. If K is a simplicial complex, the suspension of K, SK, is the simplicial complex obtained by gluing together two copies of the cone of K, CK, along the subcomplex K, $SK = CK \cup_K CK$. (If $K = (V, S)$, then SK has as its set of vertices $V \amalg \{1, -1\}$ and as its simplices, the subsets σ, $\sigma \amalg \{1\}$, and $\sigma \amalg \{-1\}$ where σ ranges over the simplices S of K.)

   (a) Use the Mayer-Vietoris sequence to construct an isomorphism $s\colon \tilde{H}_*(K) \cong \tilde{H}_{*+1}(SK)$.

   (b) Show that a simplicial map $f\colon K \to K$ extends (uniquely) to a simplicial map $Sf\colon SK \to SK$ that is the identity on 1 and -1.

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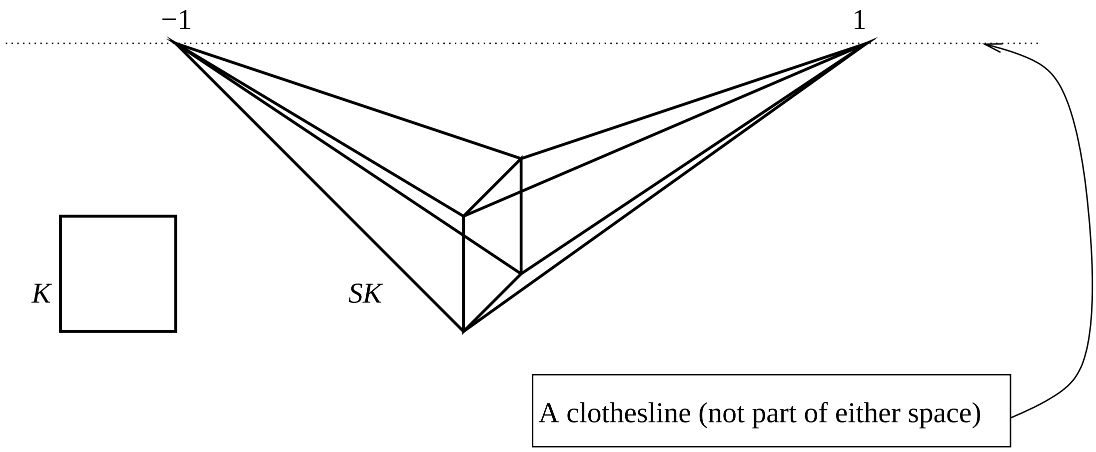
(c) Show that (in the notation of the previous parts) $s \circ \tilde{H}_* f = \tilde{H}_{*+1} Sf \circ s$. (See also 7(a) below.)

(d) Consider the simplicial map $a\colon SK \to SK$ that is the identity on K and switches 1 and -1. Show that $H_* a\colon H_* SK \to H_* SK$ is multiplication by -1.



5. The antipodal map on S^n.

(a) Let K be a simplicial complex homeomorphic to the sphere S^{n-1}. Show that the suspension of K is homeomorphic to the sphere S^n.

(b) Show that the antipodal map $S^n \to S^n$ induces on homology the map multiplication by $(-1)^{n+1}$.

(c) Show that if n is even, the antipodal map on S^n is not homotopic to the identity.

6. **The Five Lemma**

Consider the following commutative diagram, where the rows are exact.

$$\begin{array}{ccccccccc}
A & \longrightarrow & B & \xrightarrow{f} & C & \xrightarrow{g} & D & \longrightarrow & E \\
{\scriptstyle\alpha}\downarrow{\scriptstyle\cong} & & {\scriptstyle\beta}\downarrow{\scriptstyle\cong} & & {\scriptstyle\gamma}\downarrow{\scriptstyle ???} & & {\scriptstyle\delta}\downarrow{\scriptstyle\cong} & & {\scriptstyle\epsilon}\downarrow{\scriptstyle\cong} \\
A' & \longrightarrow & B' & \xrightarrow[f']{} & C' & \xrightarrow[g']{} & D' & \longrightarrow & E'
\end{array}$$

Suppose that the maps α, β, δ, and ϵ are known to be isomorphisms. The problem is to show that γ is then an isomorphism.

(a) First show that γ is a monomorphism (i.e., is injective): Take an element x in the kernel of γ. (1) Show that $g(x)$ is zero and conclude that $x = f(y)$ for some $y \in B$. (2) Show that y is in the image of A and conclude that $x = 0$.

Problem continues on the next page.

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(b) Now show that γ is an epimorphism (i.e., is surjective). Take an element $x' \in C'$ and show that it is in the image of γ as follows: (1) Show that there exists an element z in C so that $g(\delta(z)) = g'(x')$. (2) Show that there exists an element $y \in B$ so that $f'(\beta(y)) = x' - \gamma(z)$ and conclude that $\gamma(z + f(y)) = x'$.

7. Let K and K' be simplicial complexes and $A \subset K$, $A' \subset K$ subcomplexes. Let $f\colon K \to K'$ be a simplicial map that sends A into A'.

(a) Use the definition of the connecting homomorphism to show that the following diagram commutes

$$\begin{array}{ccc} H_{n+1}(K,A) & \xrightarrow{\partial} & H_n(A) \\ \downarrow{\scriptstyle H_{n+1}f} & & \downarrow{\scriptstyle H_n f} \\ H_{n+1}(K',A') & \xrightarrow[\partial]{} & H_n(A') \end{array}$$

(b) Show that if any two of the maps $H_*(A) \to H_*(A')$, $H_*(K) \to H_*(K')$, and $H_*(K,A) \to H_*(K',A')$ induced by f are isomorphisms (for all $*$), then so is the third.

(c) Suppose $K = A \cup B$ and $K' = A' \cup B'$ for subcomplexes $B \subset K$ and $B' \subset K'$ and suppose f also sends B into B'. Show that if the maps $H_*(A \cap B) \to H_*(A' \cap B')$, $H_*(A) \to H_*(A')$, and $H_*(B) \to H_*(B')$ induced by f are all isomorphisms, then so is $H_*(K) \to H_*(K')$

8. This problem studies the map $u\colon Sd\,K \to K$ defined in Example Sheet 3s, problem $7\frac{1}{2}$ for an ordered simplicial complex K. In this problem, we will show that u induces an isomorphism on homology. The proof is by double induction, over m, n: Assume that for every ordered simplicial complex K with $\dim K \leq m$ having n or fewer m-simplices, the map $u\colon Sd\,K \to K$ induces an isomorphism on homology. (Here $\dim K$ denotes the largest dimension of a simplex of K.)

(a) Check the base cases $m = 0$, n arbitrary, and (assuming the cases $m-1$ and n arbitrary) m arbitrary, $n = 0$.

(b) Recall that by definition, $Sd\,\Delta[m]$ is the cone $C\,Sd\,\partial\Delta[m]$. Prove that $u\colon Sd\,\Delta[m] \to \Delta[m]$ induces an isomorphism on homology.

(c) Complete the argument by proving the inductive step.

End of Example Sheet 4.

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