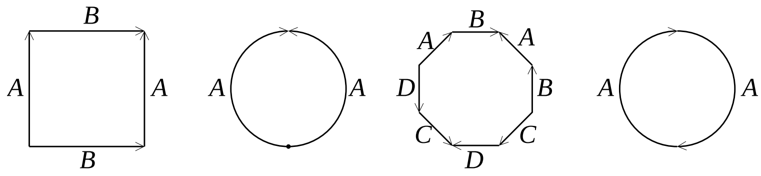
Algebraic Topology 2004 **Example Sheet 3**

1. Polygon gluing diagrams are not triangulations of the spaces. Subdivide the gluing diagrams for the torus, the sphere, the two-holed torus[1], and the projective plane to get triangulations.



2. Prove the second half of the geometric realization theorem: If X and Y have triangulations with isomorphic underlying abstract simplicial complexes, then X and Y are homeomorphic spaces.

3. A set of points $\{x_1, \ldots, x_\ell\}$ in $\mathbb{R}^m$ is said to be "in general position" if the affine span of every subset with $i + 1$ elements is i dimensional for all $i + 1 \leq m + 1$. (For example, the empty set is in general position for any $\mathbb{R}^m$; any subset of $\mathbb{R}$ is in general position; $\{(0,0),(1,0),(0,1),(1,1)\}$ is in general position in $\mathbb{R}^2$ but $\{(0,0),(1,0),(0,1),(1/2,1/2)\}$ is not.)

   (a) If $\{x_1, \ldots, x_\ell\} \subset \mathbb{R}^m$ is in general position, then the set

   $$\{x \in \mathbb{R}^m \mid \{x_1, \ldots, x_\ell, x\} \text{ is in general position}\}$$

   is an open dense subset. (Hence the name "general position".)

   (b) Show that if K is a simplicial complex of dimension n (i.e., having no $n+1$ simplices), it has a geometric realization by affine simplices in $\mathbb{R}^{2n+1}$. (Our standard model used $\mathbb{R}^v$ where v is the number of vertices in K.)

4. Show that a set of vertices in a simplicial complex forms a simplex if and only if the intersection of their open stars is non-empty in the geometric realization.

5. Use the simplicial approximation theorem to show:

   (a) If X and Y are compact triangulable spaces, then there are at most countably many homotopy classes of maps from X to Y.

   (b) If $m < n$, then any map $S^m \to S^n$ is homotopic to the constant map.

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M.A.Mandell@dpmms.cam.ac.uk
http://www.dpmms.cam.ac.uk/site2002/Teaching/II/AlgebraicTopology/

6. Let K be a simplicial complex.

(a) Show that if $|K|$ is connected, then any two vertices in K can be connected by a sequence of edges in K.

(b) Let K_2 be the "2-skeleton" of K: It is the subcomplex of K containing all vertices, 1-simplices, and 2-simplices. For a vertex a, show that $\pi_1(K_2, a) \to \pi_1(K, a)$ is an isomorphism.

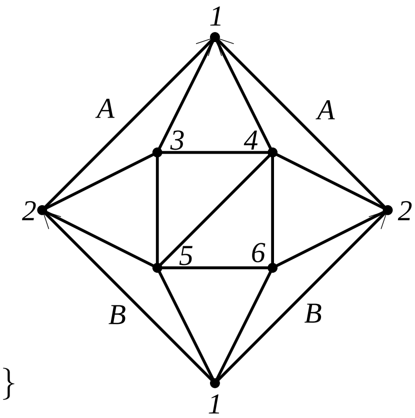
7. Use your triangulations from problem 1 to compute the homology groups of: the torus, the sphere, the two-holed torus, and the projective plane.

8. A pseudo n-manifold is a simplicial complex K with the following properties:
(i) Every simplex is a subsimplex of an n-simplex.
(ii) Every $(n-1)$-simplex is a face of exactly two n-simplices.
(iii) For any two n-simplices σ and τ, there is a sequence $\sigma = \sigma_0, \sigma_1, \dots, \sigma_r = \tau$ where each σ_i and σ_{i+1} intersect along an $(n-1)$-simplex.

(a) Show that for any triangulation of a connected 2-manifold, the simplicial complex is a pseudo 2-manifold[2]. (This holds for triangulations of connected n-manifolds as well.)

(b) Show that $V = \{1, 2, 3, 4, 5, 6\}$ and

$$S = \{ \quad \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \\ \{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{1,6\}, \\ \{2,3\}, \{2,4\}, \{2,5\}, \{2,6\}, \\ \{3,4\}, \{3,5\}, \{4,5\}, \{4,6\}, \{5,6\} \\ \{1,2,3\}, \{1,2,4\}, \{1,2,5\}, \{1,2,6\}, \\ \{1,3,4\}, \{1,5,6\} \\ \{2,3,5\}, \{2,4,6\}, \{3,4,5\}, \{4,5,6\} \quad \}$$



defines a simplicial complex that is a pseudo 2-manifold but not a 2-manifold.

(c) Show that if $H_n(K)$ is non-trivial, then $H_n(K) \cong \mathbb{Z}$ and is generated by the sum of the n-simplices with signs.

9. Assume the homology of the $(n+1)$-simplex $\Delta[n+1]$ satisfies

$$H_i(\Delta[n+1]) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & i > 0. \end{cases}$$

Use the homeomorphism $S^n \cong \partial\Delta[n+1]$ to prove

$$H_i(S^n) = \begin{cases} \mathbb{Z} & i = 0, n \\ 0 & i \neq 0, n. \end{cases}$$

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M.A.Mandell@dpmms.cam.ac.uk
http://www.dpmms.cam.ac.uk/site2002/Teaching/II/AlgebraicTopology/

10. Use the last problem to prove:

   (a) $\mathbb{R}^m$ and $\mathbb{R}^n$ are not homeomorphic if $m \neq n$.

   (b) The disk B^{n+1} does not retract onto the sphere S^n. (Recall, we showed in class that this implies the Brouwer fixed point theorem for B^{n+1}.)

11. For a compact triangulable space X, define the Euler characteristic $\chi(X)$ by

$$\chi(X) = h_0 - h_1 + h_2 - \cdots,$$

where h_i is the dimension of the real vector space $H_i(X; \mathbb{R})$. (Since $h_i = 0$ for i large, the formula above is a finite sum.)

   (a) Fix a triangulation of X, and show

$$\chi(X) = s_0 - s_1 + s_2 - \cdots,$$

   where s_i denotes the number of i-simplices in the triangulation. [Hint: In the terminology of linear algebra, the dimension of the image of a linear transformation is called the "rank" and the dimension of its kernel is the called the "nullity".]

   (b) Prove Euler's theorem: If P is a convex polyhedron in $\mathbb{R}^3$, then $F-E+V = 2$. [Hint: Put a new vertex at the center of each polygonal face.]

12. Let K be a simplicial complex and $f: K \to K$ a simplicial self-map. Choose an order for the vertices of K and let C_n denote the vector space of n-chains with coefficients in $\mathbb{R}$ (for each n) and $C_n f: C_n \to C_n$ the induced linear transformation.

   (a) Use the standard basis for C_n (of n-simplices of K) to show that if the trace $Tr(C_n f)$ is non-zero, then the geometric realization of f has a fixed point.

   (b) Show that $Tr(C_n f) = Tr(Z_n f) + Tr(B_{n+1} f)$ and $Tr(Z_n f) = Tr(B_n f) + Tr(H_n f)$, where
      - Z_n denotes the vector space of n-cycles (with coefficients in $\mathbb{R}$) and $Z_n f$ is the induced map on the cycles
      - $B_{n+1} = C_n/Z_n$ and $B_{n+1} f$ is the induced map on B_{n+1}, and
      - $H_n = H_n(K; \mathbb{R}) = Z_n/B_n$ and $H_n f$ is the induced map on H_n.

   (c) Show that

$$\sum (-1)^n Tr(C_n f) = \sum (-1)^n Tr(H_n f)$$

   (d) Conclude that if the number of the previous part is non-zero, the geometric realization of f has a fixed point.

   Problem continues on the next page.

This is a weak version of the Lefschetz fixed point theorem. It only takes uniform continuity, plus the simplicial approximation theorem, plus some bookkeeping to prove the full version:

> **Theorem** (The Lefschetz fixed point theorem) Let X be a compact triangulable space and let $f\colon X \to X$ be a continuous map. If the Lefschetz number $\Lambda(f) = \sum(-1)^n Tr(H_n f)$ is non-zero, then f has a fixed point.

(See for example Munkres, *Elements of Algebraic Topology*, pp. 125–126 for an argument.)

What makes this theorem particularly powerful is that the formula for $\Lambda(f)$ only depends on the homotopy class of the map; this is already interesting for simplicial maps. (Also note: The Euler characteristic is the Lefschetz number of the identity map.)

Endnotes

[1]The version of the example sheets handed out in class had a different gluing diagram for the two-holed torus. Although it does not change the problem, the original diagram was too ugly to leave uncorrected (and for this reason is not included here for comparison).

[2]The version of the example sheets handed out in class said "pseudo n-manifold" here; please note the correction.

End of Example Sheet 3.

M.A.Mandell@dpmms.cam.ac.uk
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