Algebraic Topology 2004 **Example Sheet 2**

1. Consider a space $X = U \cup V$ with U and V open and $U \cap V$ connected. Let $x \in U \cap V$. Label the maps on $\pi_1(-, x)$ induced by the inclusions as indicated in the diagram.

$$\begin{array}{ccc} \pi_1(U \cap V, x) & \xrightarrow{f_1} & \pi_1(U, x) \\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle g_1} \\ \pi_1(V, x) & \xrightarrow[g_2]{} & \pi_1(X, x) \end{array}$$

   (a) Show that if f_1 is an isomorphism, then so is g_2.

   (b) Show that if f_1 is an epimorphism, then so is g_2, and identify its kernel.

   (c) Given a presentation (a description in terms of generators and relations) for each of $\pi_1(U \cap V, x)$, $\pi_1(U, x)$, and $\pi_1(V, x)$, give a presentation for $\pi_1(X, x)$.

2. Let (X, x) and (Y, y) be locally contractible based spaces. Prove that $\pi_1(X \vee Y, x)$ is isomorphic to $\pi_1(X, x) * \pi_1(Y, y)$. Show that a bouquet of n circles ($S^1 \vee \cdots \vee S^1$) has fundamental group the free group on n generators.

3. Let L be the "infinite ladder of circles" given by the subset of $\mathbb{R}^2$ consisting of the union of the circle of radius $1/2$ around each point $(n, 0)$ for $n \in \mathbb{Z}$.

   (a) Choose a basepoint for L and show that the fundamental group is a free group on an infinite number of generators.

   (b) Show that the action of $\mathbb{Z}$ on L (where $n \in \mathbb{Z}$ acts by the map that sends (x, y) to $(x + n, y)$) is properly discontinuous and its quotient is homeomorphic to $S^1 \vee S^1$.

   (c) Conclude that the free group on an infinite number of generators is isomorphic to a subgroup of the free group on two generators.

4. Fundamental groups of complements.

   (a) Show that the fundamental group of a complement in $\mathbb{R}^2$ of a finite number of points is a free group.

   (b) Show that the fundamental group of a complement in $\mathbb{R}^3$ of a finite number of points is simply connected.

   (c) Show that the fundamental group of the complement in $\mathbb{R}^3$ of the circle $\{(x, y, 0) \mid x^2 + y^2 = 1\}$ is a free group on one generator.

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5. Let (X, x) be a based space, $f: S^1 \to X$ a based map, and $Y = X \cup_f B^2$ the space obtained by gluing B^2 onto X along f.

   (a) Show that if X is locally contractible then so is Y.

   (b) Let $\alpha \in \pi_1(X, x)$ be the element represented by f. Show that $\pi_1(Y, x)$ is isomorphic to $\pi_1(X, x)/N$, where N is the smallest normal subgroup of $\pi_1(X, x)$ containing α.

   (c) Show that any finitely presented group is the fundamental group of a locally contractible space: Given a presentation of the group in terms of a finite number of generators and a finite number of relations, construct a locally contractible space with isomorphic fundamental group.

6. This exercise and the next attempts to define a concept of "orientation" for a 2-manifold M (a metrizable space where every point lies in an open set that is homeomorphic to $\mathbb{R}^2$ – such an open set is called a "euclidean neighborhood").

   (a) For a point x in M, and a euclidean neighborhood E of x, show that (for any basepoint) the fundamental group of $E - \{x\}$ is a free group on one generator (i.e., is isomorphic to $\mathbb{Z}$).

   (b) Show that if $E \subset E'$ are euclidean neighborhoods of x, then the induced map $\pi_1(E - \{x\}) \to \pi_1(E' - \{x\})$ is an isomorphism.

   (c) Let E_1 and E_2 be two euclidean neighborhoods of x. Show that there is a euclidean neighborhood E_3 of x which is contained in both E_1 and E_2.

   (d) Let E_0 be a euclidean neighborhood of x, and let o_{x,E_0} be a choice of generator for $\pi_1(E_0 - \{x\})$. Show that there is one and only one way to assign a generator $o_{x,E}$ for each euclidean neihborhood E of x (with the one as given on E_0) such that whenever $E \subset E'$, the induced isomorphism of π_1 sends $o_{x,E}$ to $o_{x,E'}$. We call such a system of choices a "local orientation at x". (You can think of this as choosing a counter-clockwise direction for circles around x.)

7. Let M be a 2-manifold. Choose a local orientation around each point. Let B denote the open ball of radius 1 in $\mathbb{R}^2$.

   (a) Let U be an open set of M and $f: U \to \mathbb{R}^2$ a homeomorphism. Show that (for any basepoint) $\pi_1(U - f^{-1}B)$ is a free group on one generator

   (b) Show that for every x in $f^{-1}B)$ the inclusion of $U - f^{-1}B$ in $U - \{x\}$ induces an isomorphism on fundamental groups.

   Problem continues on the next page.

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An "orientation" for M is a choice of local orientations such that whenever U and f are as above, there exists a generator $o_{U,f}$ of $\pi_1(U - f^{-1}B)$ so that the inclusion of $U - f^{-1}B$ in $U - \{x\}$ sends $o_{U,f}$ to $o_{U,x}$. M is said to be "orientable" if there exists an orientation.

(c) Show that the local orientations form an orientation if and only if for every point x in M, there exists an open set U, a homeomorphism $f\colon U \to \mathbb{R}^2$ with $x \in f^{-1}B$, and a generator $o_{U,f}$ of $\pi_1(U - f^{-1}B)$ such that for every y in $f^{-1}B$, the inclusion of $U - f^{-1}B$ in $U - \{y\}$ sends $o_{U,f}$ to $o_{U,y}$.

8. Orientability.

(a) Show that the sphere and the torus are orientable.

(b) Show that the (open) Möbius band is not orientable.

(c) Show that if M contains a nonempty open subset that is not orientable then M is not orientable.

(d) Show that the projective plane and Klein bottle are not orientable.

9. Let M be a connected manifold. Show that M is "homogeneous" in the following sense: Given $x, y \in M$, there is a homeomorphism from M to M (an "automorphism") that sends x to y. Here is one possible outline: For each $x \in M$, let

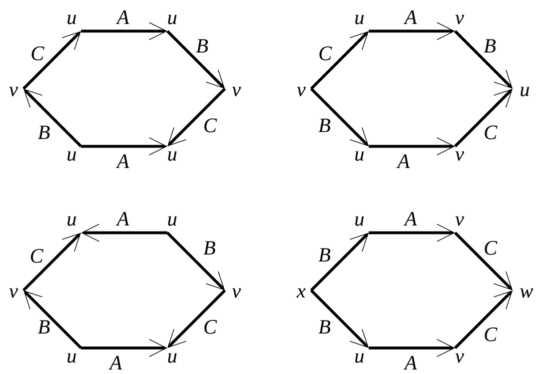
$$S_x = \{y \in M \mid \text{there exists an automorphism of } M \text{ taking } x \text{ to } y\}$$

(a) Show that S_x contains a euclidean neighborhood of x.

(b) Show that if $y \in S_x$ and $z \in S_y$ then $z \in S_x$. Conclude that S_x is open.

(c) Show that S_x is closed. [Hint: Consider a euclidean neighborhood of a limit point.]

10. Take the hexagon and glue the vertexes and sides as indicated by the labels. Which spaces are manifolds, and which manifolds are they?



End of Example Sheet 2.

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